Algebra n. 3 - NOTE ALLA LEZIONE 17

Osservazione 17.10

e) Un esempio di questo tipo è presentato nell'articolo allegato a queste note.

Esempi 17.13

a) Dato un intero positivo *n,* sia $a \in \mathbb{Z}$. Allora, nel gruppo additivo $\mathbb{Z}_n$, si ha $n[a]_n = [0]_n$. Ciò prova che l'elemento $[a]_n$ non è libero su $\mathbb{Z}$. Ciò implica che nessun sottoinsieme non vuoto di $\mathbb{Z}_n$ è libero su $\mathbb{Z}$.

b) Siano $r, s \in \mathbb{Z}$, con $s \neq 0$, e sia $\alpha = \frac{r}{s}$. Sia inoltre *p* un numero primo che non divide *s*. Allora il numero razionale $\frac{1}{p}$ non appartiene al sotto-$\mathbb{Z}-$modulo (sottogruppo) di $\mathbb{Q}$ generato da α, che è l'insieme $\{k\alpha | k \in \mathbb{Z}\}$. Infatti, se esistesse un intero *k* tale che $\frac{kr}{s} = \frac{1}{p}$, allora si avrebbe $pkr = s$, contro l'ipotesi che *p* non divida *s*. Ciò prova che nessun sottoinsieme di $\mathbb{Q}$ formato da un solo elemento genera $\mathbb{Q}$ come $\mathbb{Z}-$modulo.

Siano ora $u, v \in \mathbb{Z}$, con $v \neq 0$, e sia $\beta = \frac{u}{v}$. Supponiamo che $\alpha \neq \beta$. Si ha $us\alpha - rv\beta = 0$, ove gli interi us, rv non sono entrambi nulli (infatti *s* e *v* sono non nulli, mentre uno fra *r* ed *u* è non nullo). Ciò prova che l'insieme $\{\alpha, \beta\}$ è non libero su $\mathbb{Z}$, e tale è ogni sottoinsieme di $\mathbb{Q}$ formato da almeno due elementi.

Esempio 17.15

- Il sotto-$\mathbb{Z}-$modulo (sottogruppo) di $\mathbb{Z}$ generato da $\{2,3\}$ è $\{2a + 3b | a, b \in \mathbb{Z}\} = \mathbb{Z}$. In generale sappiamo, infatti, in base all'Esercizio 1.3, che, dati due interi coprimi m_1, m_2, $\langle m_1, m_2 \rangle = \mathbb{Z}$. Questa – ricordiamo – è immediata conseguenza del Lemma di Bézout. Dunque $\{2,3\}$ è un sistema di generatori, e come tale è minimale, in quanto i suoi sottoinsiemi propri $\varnothing, \{2\}, \{3\}$, generano, nell'ordine, i sottogruppi $\{0\}, 2\mathbb{Z}, 3\mathbb{Z}$. Non è, tuttavia, un insieme libero (e pertanto non è una base), in quanto $3 \cdot 2 - 2 \cdot 3 = 0$.
- L'insieme $\{2\}$ è libero, dato che 2 è un elemento regolare (non zero-divisore) dell'anello $\mathbb{Z}$. Ma come tale è massimale: infatti, per ogni intero *m* distinto da 2, si ha $m \cdot 2 - 2 \cdot m = 0$, da cui segue che l'insieme $\{2, m\}$ non è libero su $\mathbb{Z}$. Per il resto, abbiamo già osservato che $\{2\}$ non è un sistema di generatori per $\mathbb{Z}$, e dunque non è una base.
- L'insieme $\{6,15,10\}$ genera $\mathbb{Z}$, in quanto, per l'Esercizio 1.3, si ha $\langle 6,10,15 \rangle = \langle \mathrm{MCD}(6,10,15) \rangle = \langle 1 \rangle = \mathbb{Z}$. La minimalità risulta dalle seguenti osservazioni:

$\langle 6,10\rangle = 2\mathbb{Z},$
$\langle 6,15\rangle = 3\mathbb{Z},$
$\langle 10,15\rangle = 5\mathbb{Z}.$
In realtà, è possibile costruire sistemi minimali di generatori di $\mathbb{Z}$ di qualsiasi cardinalità finita *n*. Per $n=1$ si ha $\{1\}$, per ogni $n>1$, dati *n* numeri primi positivi distinti $p_1,\dots,p_n$, si ponga, per ogni indice *i*, $m_i = \prod_{\substack{j=1\\ j\neq i}}^{n} p_j$. Allora $\mathrm{MCD}(m_1,\dots,m_n)=1$, (e dunque $\langle m_1,\dots,m_n\rangle = \mathbb{Z}$), mentre, per ogni indice *i*, $\langle m_1,\dots,\hat{m}_i,\dots,m_n\rangle = \langle \mathrm{MCD}(m_1,\dots,\hat{m}_i,\dots,m_n)\rangle = p_i\mathbb{Z} \neq \mathbb{Z}.$

Osservazione 17.18

Ogni elemento dell'ideale $([2]_6)$ è annullato da $[3]_6$.

Teorema 17.19

Illustriamo l'enunciato con un esempio semplicissimo. Consideriamo $M=\mathbb{Z}, N=2\mathbb{Z}.$ Allora $n=m=1,$ e si può prendere $a_1=1, b_1=2,$ così che $T=(2).$ E, in effetti, $(\mathbb{Z}:2\mathbb{Z})=2.$

Dimostrazione della **Proposizione 17.14**:

Per ogni $m\in M$, esistono $a_1,\dots,a_n\in A$ tali che $m=\sum_{i=1}^{n} a_i x_i$. Allora

$$IM + m = \sum_{i=1}^{n} IM + a_i x_i = \sum_{i=1}^{n} (I+a_i)(IM+x_i).$$

Ciò prova che l'insieme $\pi(B)=\{IM+x_1,\dots,IM+x_n\}$ è un sistema di generatori di M/IM su A/I.

Sia ora $y\in IM$. Allora esistono $c_1,\dots,c_r\in I$ e $y_1,\dots,y_r\in M$ tali che $y=\sum_{i=1}^{r} c_i y_i$. D'altra parte, per ogni $i=1,\dots,r,$ esistono $d_{i1},\dots,d_{in}\in A$ tali che $y_i=\sum_{j=1}^{n} d_{ij}x_j$. Ne consegue che

$$y=\sum_{i=1}^{r} c_i\left(\sum_{j=1}^{n} d_{ij}x_j\right)=\sum_{j=1}^{n}\left(\sum_{i=1}^{r} c_i d_{ij}\right)x_j,$$

ove, per ogni *j*, $\sum_{i=1}^{r} c_i d_{ij} \in I.$

Maximal Sets of Linearly Independent Vectors in a Free Module Over a Commutative Ring

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ABSTRACT

The following theorem is proved: If R is a noetherian ring, M is a free R-module of rank n, and $\{v_1,\dots,v_s\}$ is the maximal set of linearly independent vectors, then always $s=n$. An example is also given of a commutative ring R for which the above theorem is false for every $n \geqslant 2$.

1. INTRODUCTION

Let R be a commutative ring, and let $M \cong R^n$ be a free R-module. We will say that vectors $\{v_1,\dots,v_s\}$ are linearly independent iff the following implication holds:

$$\text{if} \quad \sum \lambda_i v_i = 0 \quad \text{for } \lambda_i \in R, \quad \text{then} \quad \lambda_i = 0 \quad \text{for } i = 1,\dots,s.$$

The set $\{v_1,\dots,v_s\}$ of linearly independent vectors will be called maximal if for every $w \in M$ the vectors v_i, $i=1,\dots,s$, and w are not linearly independent.

Let us assume that $\{v_1,\dots,v_s\} \subset M$ is the maximal set of linearly independent vectors of M. It is natural to try to relate s to n. It is well known that always $s \leqslant n$. In this note we prove

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THEOREM. *If R is a noetherian ring, M is a free R module of rank n, and* $\{v_1, \ldots, v_s\}$ *is the maximal set of linearly independent vectors, then always* $s = n$.

There is also given an example of a commutative ring R for which the above theorem is false for every $n \geqslant 2$.

2. PROOF OF THE THEOREM

The following theorem is well known (see [2, Theorem 3.5.1, p. 124] or, for an equivalent version, [3, Proposition 12, p. 519]):

THEOREM of McCoy. *Let R be a ring. A system of n homogeneous linear equations in p unknowns,*

$$\sum_{j=1}^{p} \alpha_{ij} x_j = 0, \quad i = 1, \ldots, n, \qquad \alpha_{ij} \in R,$$

has a nontrivial solution in R if and only if either $p > n$ *or* $p \leqslant n$ *and there exists a nonzero element in R annihilating all the* $p \times p$ *minors of the matrix* $[\alpha_{ij}]$.

REMARK. It easily follows from this theorem that if $\{v_1, \ldots, v_s\}$ is the set of linearly independent vectors of a free R-module $M \cong R^n$, then always $s \leqslant n$.

THEOREM. *Let R be a noetherian ring, and M be a free R-module of rank n. Let* $\{v_1, \ldots, v_s\}$ *be the maximal set of linearly independent vectors of M. Then* $s = n$.

Proof. The proof is based on some lemmas.

LEMMA 1 (See [2, p. 108]). *Let* $P_1, \ldots, P_k$ *be prime ideals of a ring R, let I be an ideal of R, and let x be an element of R such that the ideal* $\langle I, x \rangle$ *is not contained in* $P_1 \cup \cdots \cup P_k$. *Then there exists an element* $y \in I$ *such that* $x + y \notin P_1 \cup \cdots \cup P_k$.

LEMMA 2. *Let* $\mathbf{t} = (t_1, \ldots, t_n) \in R^n$ *be a linearly independent vector. There exists an R-automorphism* $A : R^n \to R^n$ *which has the form* $A((x_1, \ldots, x_n)) = (x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n, x_2, \ldots, x_n)$, $\lambda_i \in R$ *for* $i = 2, \ldots, n$, *such that* $A(\mathbf{t}) = (t'_1, t_2, \ldots, t_n)$ *and* t'_1 *is not a zero divisor.*

Proof. Let $z(R)$ denote the set of all zero divisors of R. It is well known (see e.g. [1, Propositions 4.7, 7.17]) that $z(R) = P_1 \cup \cdots \cup P_k$, where P_i, $i = 1, \ldots, k$, are all associated primes of (0) and there exist elements $x_i \neq 0$, $i = 1, \ldots, k$, $x_i \in R$, such that $P_i = \mathrm{Ann}(x_i) = \{r \in R : rx_i = 0\}$. Let q denote the ideal $\langle t_1, \ldots, t_n \rangle$. The ideal q is not contained in $z(R)$. Indeed, if $q \subset z(R)$, then there exists such i that $q \subset P_i$ (see e.g. [1, Proposition 1.11]), and it follows from the above that $x_i q = (0)$. Hence $x_i \mathbf{t} = \mathbf{0}$—we get a contradiction with our assumption that $\mathbf{t}$ is linearly independent. Let us put $x = t_1$, $I = \langle t_2, \ldots, t_n \rangle$ and apply Lemma 1. We see at once that there exist elements $\lambda_2, \ldots, \lambda_n$ belonging to R such that $t_1 + \sum_{i=2}^{n} \lambda_i t_i \notin z(R)$. Now it is enough to put

$$A((x_1, \ldots, x_n)) = \left(x_1 + \sum_{i=2}^{n} \lambda_i x_i, x_2, \ldots, x_n \right).$$

This completes the proof. ∎

LEMMA 3. *Let* α_i, $i = 1, \ldots, s$, *be elements of R none of which are zero divisors. Let* $\mathbf{v}_1, \ldots, \mathbf{v}_s$ *be vectors of M. Then the set* $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ *is the maximal set of linearly independent vectors if and only if the set* $\{\alpha_1 \mathbf{v}_1, \ldots, \alpha_s \mathbf{v}_s\}$ *is the maximal set of linearly independent vectors.*

Proof. $\Rightarrow$: The vectors $\alpha_i \mathbf{v}_i$, $i = 1, \ldots, s$, are linearly independent. Indeed, if λ_i, $i = 1, \ldots, s$, are elements of R such that

$$\sum \lambda_i (\alpha_i \mathbf{v}_i) = \mathbf{0},$$

then $\lambda_i \alpha_i = 0$ for $i = 1, \ldots, s$ and hence $\lambda_i = 0$ for $i = 1, \ldots, s$. To prove that the set $\{\alpha_i \mathbf{v}_i, i = 1, \ldots, s\}$ is maximal, suppose that there exists a vector $\mathbf{w}$ such that the vectors $\alpha_i \mathbf{v}_i$, $i = 1, \ldots, s$, and $\mathbf{w}$ are linearly independent. Then the vectors $\mathbf{v}_i$, $i = 1, \ldots, s$, and $\mathbf{w}$ are linearly independent—to see that, it is enough to multiply the equality

$$\sum_{i=1}^{n} \lambda_i \mathbf{v}_i + \lambda \mathbf{w} = \mathbf{0}$$

by $\alpha = \alpha_1 \cdots \alpha_s \notin z(R)$. But the set $\{v_i,\ i = 1,\dots,s\}$ is the maximal set—a contradiction. This completes the proof of this implication.

$\Leftarrow$: Let us assume that the set $\{\alpha_i v_i,\ i = 1,\dots,s\}$ is the maximal set of linearly independent vectors. We will show that the set $\{v_i,\ i = 1,\dots,s\}$ is also maximal. By multiplying the equality

$$\sum_{i=1}^{n} \lambda_i v_i = 0$$

by α (as above) we easily obtain that the vectors v_i, $i = 1,\dots,s$, are linearly independent. Let us assume that the vectors v_i, $i = 1,\dots,s$, and w are linearly independent. Then the vectors $\alpha_i v_i$, $i = 1,\dots,s$, and w are linearly independent, as is obvious. This is a contradiction, which finishes the proof. ∎

LEMMA 4. *Let* $v_1,\dots,v_s$ *be vectors of* M, *and let* $w_1 = v_1$, $w_i = v_i - \alpha_i v_1$ *for* $i = 2,\dots,s$, $\alpha_i \in R$. *Then the set* $\{v_1,\dots,v_s\}$ *is the maximal set of linearly independent vectors if and only if the set* $\{w_1,\dots,w_s\}$ *is the maximal set of linearly independent vectors.*

Proof. It is enough to prove that for every $w \in M$ the vectors $\{v_1,\dots,v_s,w\}$ are linearly independent iff the vectors $\{w_1,\dots,w_s,w\}$ are linearly independent.

$\Rightarrow$: Let us assume that vectors $\{v_1,\dots,v_s,w\}$ are linearly independent. We will show that vectors $\{w_1,\dots,w_s,w\}$ are linearly independent. Indeed if

$$\sum_{i=1}^{n} \lambda_i w_i + \lambda w = 0,$$

then

$$\left(\lambda_1 - \sum_{i=2}^{n} \alpha_i \lambda_i\right) v_1 + \sum_{i=1}^{n} \lambda_i v_i + \lambda w = 0.$$

By our assumption we have first $\lambda = 0$ and $\lambda_i = 0$ for $i \geqslant 2$, and so $\lambda_1 = 0$.

$\Leftarrow$: The proof is exactly the same as above. ∎

Now we are in a position to prove our theorem. Let $\{v_1,\dots,v_s\}$ be the maximal set of linearly independent vectors of M. We know that $s \leqslant n$; hence it is enough to prove that $s \geqslant n$. Let $v_j = (a_{1j},\dots,a_{nj})$ for $j = 1,\dots,n$. By Lemma 2 we can change coordinates of M [by a linear automorphism A

which has the form $A((x_1,\dots,x_n)) = (x_1 + \sum_{i=2}^{n} \lambda_i x_i, x_2, \dots, x_n)$] in such a way that $a_{11} \notin z(R)$. It is clear by Lemma 3 and Lemma 4 that we can replace the vectors $v_1,\dots,v_s$ by the vector v_1 and the vectors $a_{11}v_i - a_{1i}v_1$, $i = 2,\dots,s$. Therefore we can assume that

$$\begin{aligned} v_1 &= (a_{11},\dots,a_{n1}), \qquad a_{11} \notin z(R),\\ v_2 &= (0, a_{22},\dots,a_{n2}),\\ &\vdots\\ v_s &= (0, a_{2s},\dots,a_{ns}). \end{aligned}$$

In the same manner as above [applying an automorphism of the form $A((x_1,\dots,x_n)) = (x_1, x_2 + \sum_{i=3}^{n} \lambda_i x_i, x_2,\dots,x_n)$], we can also assume that $a_{22} \notin z(R)$. Applying our Lemmas 3 and 4 twice, we can write

$$\begin{aligned} v_1 &= (a_{11},\dots,a_{n1}), \qquad a_{11} \notin z(R),\\ v_2 &= (0, a_{22},\dots,a_{n2}), \qquad a_{22} \notin z(R),\\ v_3 &= (0,0, a_{33},\dots,a_{n3}),\\ &\vdots\\ v_s &= (0,0, a_{3s},\dots,a_{ns}). \end{aligned}$$

We continue in this fashion to obtain

$$\begin{aligned} v_1 &= (a_{11},\dots,a_{n1}), & a_{11} &\notin z(R),\\ v_2 &= (0, a_{22},\dots,a_{n2}), & a_{22} &\notin z(R),\\ v_3 &= (0,0, a_{33},\dots,a_{n3}), & a_{33} &\notin z(R),\\ &\vdots & &\vdots\\ v_s &= (0,0,\dots,0, a_{ss}, a_{s+1s},\dots,a_{ns}), & a_{ss} &\notin z(R). \end{aligned}$$

Now it is obvious that $s \geqslant n$. Indeed in the opposite case let w denote the vector $(a_{1s+1},\dots,a_{ns+1})$, where $a_{js+1} = 1$ for $j = s+1$ and $a_{js+1} = 0$ for

$j \neq s+1$. It is clear by the Theorem of McCoy that the vectors $\{v_1, \ldots, v_s, w\}$ are linearly independent, because the matrix $[a_{ij}]$, $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant s+1$ has $(s+1) \times (s+1)$ minor equal to $a_{11}a_{22} \cdots a_{ss} \notin z(R)$. This is a contradiction with our assumption that the set $\{v_1, \ldots, v_s\}$ is the maximal set of linearly independent vectors. Hence we must have $s \geqslant n$. The proof is completed. ■

COROLLARY 1. *Let R be a noetherian ring, and let M be a free R-module of rank n. Let $\{v_1, \ldots, v_s\}$ be a sequence of linearly independent vectors of M. Then there exist vectors $\{w_1, \ldots, w_{n-s}\}$ of M such that the vectors $\{v_1, \ldots, v_s, w_1, \ldots, w_{n-s}\}$ are linearly independent.*

COROLLARY 2. *Let M be as above, and N be a free submodule of M. The quotient module M/N is a torsion module if and only if* rank $M =$ rank N.

3. EXAMPLE

In this section we will give an example of a commutative ring R for which our theorem is false for every $n \geqslant 2$.

Let $S = \mathbb{Z}[a, b, x_1, x_2, \ldots]$ be the ring of polynomials in infinitely many indeterminates $a, b, x_1, x_2, \ldots$. The set S is countable, so the set $S \times S$ is too. Hence $S \times S = \{(\phi_i, \psi_i),\ i = 1, 2, \ldots\}$. Let I_0 denote the ideal generated by the polynomials $(a\phi_i + b\psi_i)x_i$, $i = 1, 2, \ldots$. Let $I = \{\lambda \in S : \lambda a^n \in I_0$ and $\lambda b^n \in I_0$ for $n \gg 0\}$.

PROPOSITION 1.

(1) *$x_i \notin I$ for every $i = 1, 2, \ldots$.*
(2) *If $\lambda \in S$ and $\lambda a, \lambda b \in I$, then $\lambda \in I$.*

Proof. (1): Suppose statement (1) is false, i.e., there exists such i that $x_i \in I$. By definition of I we have $x_i a^n \in I_0$, $x_i b^n \in I_0$ for some $n \gg 0$. Hence

$$x_i a^n = \sum_{j=1}^{\infty} f_j(a\phi_j + b\psi_j)x_j,$$

where $f_j \in S$ and almost all f_j are equal to 0. There exists m such that we can consider the above equation in the ring $\mathbb{Z}[a, b, x_1, \ldots, x_m] \cong$

$\mathbb{Z}[a,b][x_1,\dots,x_m]$. Let $a\phi_i + b\psi_i = c_0 + \sum_{|\alpha|>0} c_\alpha x^\alpha$, where $c_\alpha \in \mathbb{Z}[a,b]$. Observe that $c_0 = 0$ or $c_0 \notin \mathbb{Z}$. Let $f_i = d_0 + \sum_{|\alpha|>0} d_\alpha x^\alpha$, $d_\alpha \in \mathbb{Z}[a,b]$. We have

$$a^n x_i = c_0 d_0 x_i + \sum_{|\alpha|>1} w_\alpha x^\alpha + \sum_{j\neq i} P_j x_j, \qquad w_\alpha \in \mathbb{Z}[a,b],$$

$P_j \in \mathbb{Z}[a,b][x_1,\dots,x_m]$; therefore $a^n = c_0 d_0$ and $c_0 \notin \mathbb{Z}$ (because $c_0 \neq 0$). Therefore we have $c_0 = ra^s$, where $r,s \in \mathbb{Z}$, $r \neq 0$, and $s \geqslant 1$. Now, let us consider the equality

$$x_i b^n = \sum_{j=1}^{\infty} g_j(a\phi_j + b\psi_j)x_j.$$

In the same manner as above, we can see that $c_0 = r'b^{s'}$, where $r', s' \in \mathbb{Z}$, $r' \neq 0$, and $s' \geqslant 1$. Hence

$$c_0 = ra^s = r'b^{s'},$$

which is a contradiction.

(2): Let us assume that $\lambda a, \lambda b \in I$ for some $\lambda \in S$. There exist m, n such that $(\lambda a)a^m \in I_0$, $(\lambda b)b^n \in I_0$. Let us put $s = \max(m+1, n+1)$. We have $\lambda a^s \in I_0$, $\lambda b^s \in I_0$, and by definition of I, $\lambda \in I$. ■

Proposition 2. *Let R denote the quotient ring S/I, and let $p: S \to R$ be the canonical projection. Let us denote $p(x)$ by $x^\wedge$. Then*

(1) $\langle a^\wedge, b^\wedge\rangle \subseteq z(R)$;
(2) *for every $\lambda^\wedge \in R$, if $\lambda^\wedge \neq 0$ then $\lambda^\wedge a^\wedge \neq 0$ or $\lambda^\wedge b^\wedge \neq 0$.*

Proof. (1): Let $x^\wedge \in \langle a^\wedge, b^\wedge\rangle$. Then there exists i such that $x^\wedge = a^\wedge \phi_i^\wedge + b^\wedge \psi_i^\wedge$ and $x^\wedge x_i^\wedge = 0$. Hence $x^\wedge \in z(R)$, because by Proposition 1, $x_i^\wedge \neq 0$.

(2): Let $\lambda^\wedge \in R$ and $\lambda^\wedge a^\wedge = 0$, $\lambda^\wedge b^\wedge = 0$. Then $\lambda a \in I$, $\lambda b \in I$, and by Proposition 1, $\lambda \in I$; hence $\lambda^\wedge = 0$. ■

Corollary. *Let R be a ring as above. For every $n \geqslant 2$ there exists a maximal set $\{v_1,\dots,v_s\}$ of linearly independent vectors of R^n for which $s < n$.*

Proof. Let us take $v_1 = (a^\wedge, b^\wedge, 0, \ldots, 0)$. Let v_i denote the vector with 1 in entry $i+1$ and 0 elsewhere, for $i = 2, \ldots, n-1$. Among all $(n-1)\times(n-1)$ minors of the matrix

$$\begin{bmatrix} a^\wedge & 0 & \cdots & 0 \\ b^\wedge & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

there are minors equal to $a^\wedge$ and $b^\wedge$ respectively, and by the Theorem of McCoy and Proposition 2 (2), it is easily seen that the vectors $v_1, \ldots, v_{n-1}$ are linearly independent. For every $w = (w_1, \ldots, w_n) \in R^n$ we have

$$\det \begin{bmatrix} a^\wedge & 0 & \cdots & 0 & w_1 \\ b^\wedge & 0 & \cdots & 0 & w_2 \\ 0 & 1 & \cdots & 0 & \cdot \\ 0 & 0 & \cdots & 0 & \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & w_n \end{bmatrix} \in \langle a^\wedge, b^\wedge \rangle \subseteq z(R),$$

and by the Theorem of McCoy, the vectors $v_1, \ldots, v_{n-1}, w$ are not linearly independent. Hence the set $\{v_1, \ldots, v_{n-1}\} \subset R^n$ is the maximal set of linearly independent vectors of R^n. ■

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Received 6 November 1990; final manuscript accepted 28 May 1991