MSc Lecture Course, Winter Term 2019/2020

Elliptic operators and topology

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Course description: The celebrated index theorem of Atiyah and Singer provides an explicit geometric formula for the Fredholm index of an elliptic operator on a closed manifold (the Fredholm index is the dimension of the kernel minus the dimension of the co-kernel). Proved by Atiyah and Singer for the first time in 1963, it is considered a fundamental result of the mathematics of the 20th century because of the interplay between geometry, topology, and partial differential equations.
The aim of the course is to develop the mathematical language needed to understand the Atiyah–Singer index formula and some of its main examples and applications.
Our focus will be on Dirac type operators and their squares, the generalised Laplacians; for these operators we will develop the theory of elliptic regularity. This analysis will lead us to discuss also other important results as the Hodge theorem on closed Riemannian manifolds, which shows that in every de Rham cohomology class one can always find a unique harmonic representative.
There are nowadays several ways to prove the Atiyah–Singer index theorem. We will present the ingredients of the 'heat equation proof'. The course mainly follows the presentation given in the book by John Roe.

Program outline: Short review of tools from differential geometry. Characteristic classes. Linear differential operators on manifolds. Ellipticity. Dirac type operators. The Euler–de Rham and the signature operators. Generalised Laplacians. Analysis of Dirac operators (Sobolev spaces, Sobolev embedding, elliptic estimates). Spectral properties of Dirac operators. Hodge theorem. Fredholm operators and the index problem. The heat operator. Trace class operators. The McKean–Singer formula. The heat kernel and its asymptotics.

Prerequisites: Differential geometry (manifolds, vector bundles, connections, curvature, metrics).

Date and Place: Tuesday 8:00–10:00, H4 (Geomatikum)
Thursday 12:00–14:00, H4 (Geomatikum)

Literature:

- John Roe, *Elliptic operators, topology and asymptotic methods*, second ed., Pitman Research Notes in Mathematics Series, vol. 395, Longman, Harlow, 1998.
- Blain Lawson, Marie-Louise Michelshon, *Spin Geometry*, Princeton Mathematical Series, Vol 38.

More references and lecture notes will be indicated during the lectures.