PRECORSO 8

mercoledì 17 settembre 2025 09:07

Nell'ultima lezione abbiamo discusso le principali proprietà dei logaritmi e visto qualche esempio di equazioni esponenziali

ESEMPIO

1) $\log_3(2^{10} \cdot 3^{14}) = \log_3 2^{10} + \log_3 3^{14}$

$$= 10 \log_3 2 + 14$$

2) $\log_2(2^4 + 2^5) = \log_2(2^4(1+2))$

$$= \log_2 2^4 + \log_2 3$$

$$= 4 + \log_2 3 .$$

3) $3^x = 4^{x^2}$

$\log_4 3^x = \log_4 4^{x^2}$

$x \log_4 3 = x^2$

$x^2 - x\log_4 3 = 0$

$x(x - \log_4 3) = 0$

$x = 0 \quad \vee \quad x = \log_4 3$

4) $e^{2x} + e^x - 2 = 0$

$t = e^x$

$t^2 + t - 2 = 0$

$$t_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \begin{cases} 1 \\ -2 \end{cases}$$

$t = 1 \quad \vee \quad t = -2$

$e^x = 1 \quad \vee \quad e^x = -2$ (impossibile)

$e^x = 1$

$x = \ln 1$

$x = 0.$

Unica soluzione: $x = 0.$

5) $32 \log_4^3 x - 6 \log_4 x - 1 = 0$

$\quad \| $

$(\log_4 x)^3$

c.e: $x > 0.$

$t = \log_4 x$

$32 t^3 - 6t - 1 = 0$

$t = \frac{1}{2}$ e' una soluzione:

$32 \cdot \frac{1}{2^3} - 6 \cdot \frac{1}{2} - 1 = 4 - 3 - 1 = 0$

Ruffini:

$(t - \frac{1}{2})(32t^2 + 16t + 2)$

$2(t - \frac{1}{2})(16t^2 + 8t + 1)$

$2(t - \frac{1}{2})(4t + 1)^2$

$$t = \frac{1}{2} \quad \vee \quad t = -\frac{1}{4}$$

$$\log_4 x = \frac{1}{2} \quad \vee \quad \log_4 x = -\frac{1}{4}$$

$$x = 4^{\frac{1}{2}} \quad \vee \quad x = 4^{-\frac{1}{4}}$$

$$x = 2 \qquad\qquad x = \frac{1}{\sqrt{2}}$$

Metodi simili si usano per le disequazioni (ma bisogna distinguere le basi >1 da quelle in $]0,1[$)

1) Sia $a \in \mathbb{R}$, $a > 1$. Allora:

$$x \le y \iff a^x \le a^y$$
$$x \ge y \iff a^x \ge a^y$$
$$x < y \iff a^x < a^y$$
$$x > y \iff a^x > a^y$$

Se inoltre $x, y > 0$

$$x \le y \iff \log_a x \le \log_a y$$
$$x \ge y \iff \log_a x \ge \log_a y$$
$$x > y \iff \log_a x > \log_a y$$
$$x < y \iff \log_a x < \log_a y$$

2) Sia $a \in \mathbb{R}$, $0 < a < 1$, allora:

$$x \le y \iff a^x \ge a^y$$
$$x \ge y \iff a^x \le a^y$$
$$x > y \iff a^x < a^y$$
$$x < y \iff a^x > a^y$$

Se $x, y > 0$:

$$x \le y \iff \log_a x \ge \log_a y$$
$$x \ge y \iff \log_a x \le \log_a y$$
$$x > y \iff \log_a x < \log_a y$$
$$x < y \iff \log_a x > \log_a y$$

ESEMPI

1) $$2^{x^2-4} - 3 \le 0$$

$$2^{x^2-4} \le 3$$

$$x^2 - 4 \le \log_2 3$$

$$x^2 \le \underbrace{4 + \log_2 3}_{>0}$$

$$|x| \le \sqrt{4 + \log_2 3}$$

$$-\sqrt{4+\log_2 3} \le x \le \sqrt{4 + \log_2 3}$$

2) $$\log_{\frac{1}{2}} (x + 3) \ge 3$$

c.e: $x + 3 > 0$

$$\log_{\frac{1}{2}} (x + 3) \ge 3$$

$$x + 3 \le \left(\frac{1}{2}\right)^3$$

$\frac{1}{2} < 1$ quindi va invertito il verso della disuguaglianza

$$x + 3 \le \frac{1}{8}$$

$$x \le \frac{1}{8} - 3$$

$$x \le -\frac{23}{8}$$

$$\begin{cases} x+3>0 \\ x \leq -\frac{23}{8} \end{cases} \qquad \begin{cases} x>-3 \\ x \leq -\frac{23}{8} \end{cases}$$

$$-3 < x \leq -\frac{23}{8}$$

3) $$\frac{1}{\log_5(x-1)} \leq -1$$

- c.e: $x > 1$

  $\log_5(x-1) \neq 0 \iff x-1 \neq 5^0 \iff x \neq 2$

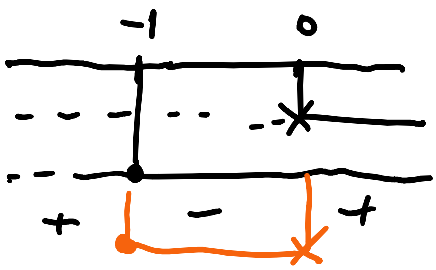
- $t = \log_5(x-1)$

  $$\frac{1}{t} \leq -1$$

$$\frac{1}{t} + 1 \leq 0$$

$$\frac{1+t}{t} \leq 0$$

$$-1 \leq t < 0$$



$$-1 \leq \log_5(x-1) < 0$$

$$5^{-1} \leq x-1 < 1$$

$$\frac{1}{5} \leq x-1 < 1 \qquad \left(\begin{cases} \frac{1}{5} \leq x-1 \\ x-1 < 1 \end{cases} \right)$$

$$\frac{6}{5} \leq x < 2$$

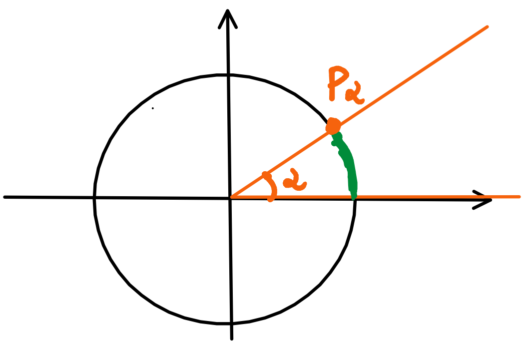
$$\begin{cases} \frac{6}{5} \le x < 2 \\ x > 1 \quad \wedge \; x \neq 2 \end{cases} \iff \frac{6}{5} \le x < 2.$$

Cenni di trigonometria

Chiamiamo CIRCONFERENZA GONIOMETRICA in $\mathbb{R}^2$ la circonferenza di centro $(0,0)$ e raggio 1 (equazione: $x^2 + y^2 = 1$)

Sulla circonferenza goniometrica è facile rappresentare gli angoli: basta tracciare una semiretta e considerare

l'angolo che essa forma con il semiasse orizzontale positivo.

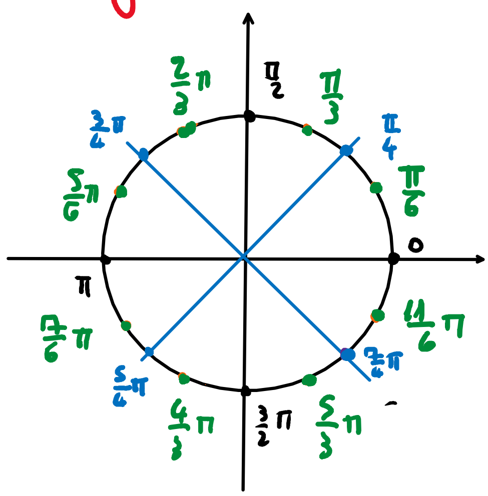


Ad ogni angolo corrisponde un arco sulla circonferenza goniometrica.
La lunghezza dell'arco associato ad α si dice MISURA IN RADIANTI DI α

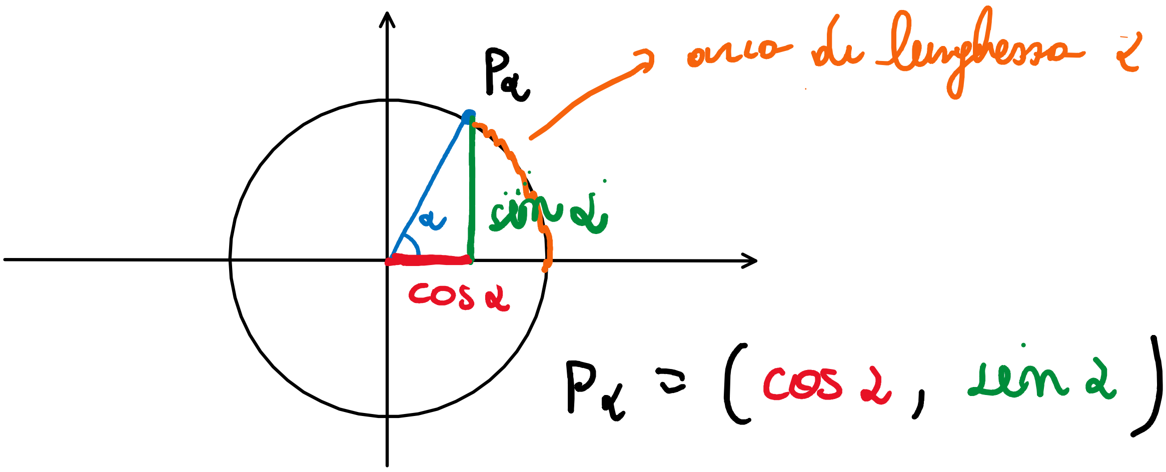
Conversione fra gradi e radianti

Gradi	Radianti
0°	0
360°	2π
180°	π
90°	$\frac{\pi}{2}$
45°	$\frac{\pi}{4}$
30°	$\frac{\pi}{6}$
60°	$\frac{\pi}{3}$
135°	$\frac{\pi}{2}+\frac{\pi}{4}=\frac{3}{4}\pi$

→ per definizione di π circonferenza goniometrica ha lunghezza 2π)



Def Sia $\alpha \in \mathbb{R}$. Sia P_α il punto della circ. goniometrica ottenuto a partire da $(1,0)$ ruotando di un angolo di ampiezza in radianti pari ad α. Le coordinate di P_α si dicono COSENO e SENO di α.



$$P_\alpha = (\cos\alpha, \sin\alpha)$$

Tabella di valori noti di seno e coseno

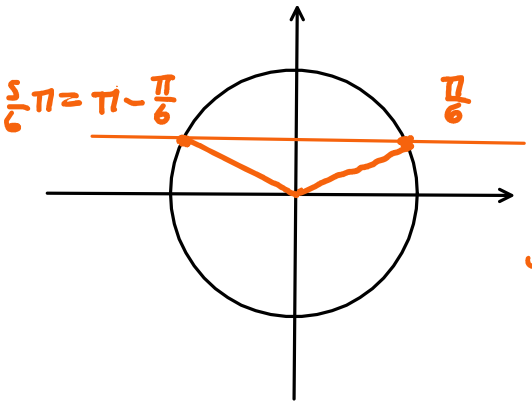
α	$\cos\alpha$	$\sin\alpha$
0	1	0
$\frac{\pi}{2}$	0	1
π	-1	0
$\frac{3}{2}\pi$	0	-1
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
$\frac{3}{4}\pi$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{5}{4}\pi$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
$\frac{7}{4}\pi$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

A partire da questa tabella si possono ricavare i valori di seno e coseno di altri angoli sfruttando le proprietà di simmetria della circonferenza. Ad esempio

$$\frac{5}{6}\pi = \pi - \frac{\pi}{6}$$

$$\cos\left(\frac{5}{6}\pi\right) = -\frac{\sqrt{3}}{2}$$

$$\sin\left(\frac{5}{6}\pi\right) = \frac{1}{2}$$



I punti che corrispondono a $\frac{\pi}{6}$ e $\frac{5}{6}\pi$ hanno la stessa coordinata y e coordinata x opposta.

Curiosità "regola del 4":

Ordinando in maniera crescente gli angoli $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ i valori del loro coseno (risp. seno) si ottengono elencando in maniera decrescente (risp. crescente) i numeri naturali ≤ 4, dividendoli per 4 e facendone la radice quadrata:

α	$\cos\alpha$	$\sin\alpha$
0	$\sqrt{\frac{4}{4}} = 1$	$\sqrt{\frac{0}{4}} = 0$
$\frac{\pi}{6}$	$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$	$\sqrt{\frac{1}{4}} = \frac{1}{2}$
$\frac{\pi}{4}$	$\sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}$	$\sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\sqrt{\frac{1}{4}} = \frac{1}{2}$	$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	$\sqrt{\frac{0}{4}} = 0$	$\sqrt{\frac{4}{4}} = 1$

PROPRIETÀ

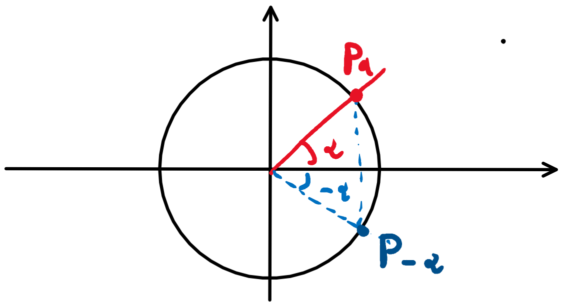
1) $\forall \alpha \in \mathbb{R}$: $\cos^2\alpha + \sin^2\alpha = 1$

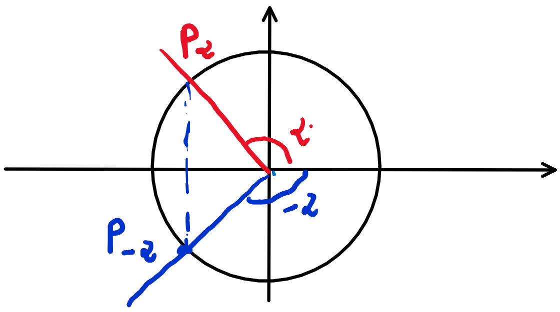
2) $\forall \alpha \in \mathbb{R}$: $|\cos\alpha| \leq 1$ $\quad |\sin\alpha| \leq 1$

$(-1 \leq \cos\alpha \leq 1 \quad -1 \leq \sin\alpha \leq 1)$

3) $\forall \alpha \in \mathbb{R}: \quad \cos(-\alpha) = \cos\alpha$

$\sin(-\alpha) = -\sin\alpha$



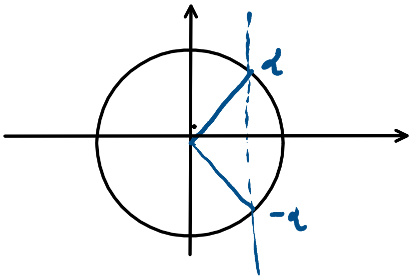


4) $\forall \alpha, \beta \in \mathbb{R}: \quad P_\alpha = P_\beta \iff \alpha = \beta + 2k\pi$ con $k \in \mathbb{Z}$

5) $\forall \alpha, \beta \in \mathbb{R}:$

$\cos\alpha = \cos\beta \iff \alpha = \beta + 2k\pi$ con $k \in \mathbb{Z}$

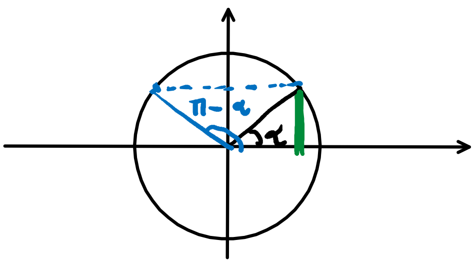
$\vee\ \alpha = -\beta + 2k\pi$ con $k \in \mathbb{Z}$



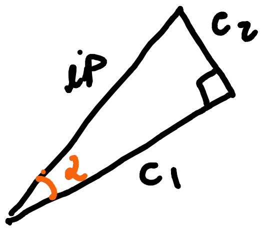
6) $\forall \alpha, \beta \in \mathbb{R}:$

$\sin\alpha = \sin\beta \iff \alpha = \beta + 2k\pi$ con $k \in \mathbb{Z}$

$\vee\ \alpha = \pi - \beta + 2k\pi$ con $k \in \mathbb{Z}$



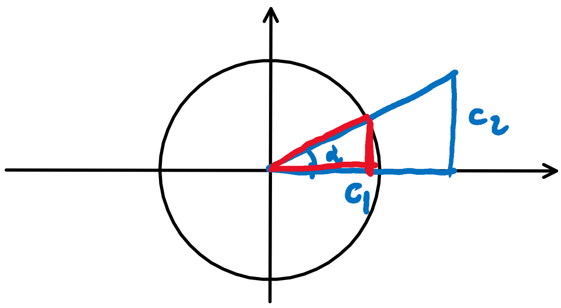
Triangolo rettangolo:



ip IPOTENUSA

c_1 CATETO ADIACENTE AD α

c_2 CATETO OPPOSTO AD α



$$\frac{ip}{1} = \frac{c_1}{\cos\alpha}$$

$$\frac{ip}{1} = \frac{c_2}{\sin\alpha}$$

cioè

$$c_1 = ip \cdot \cos\alpha$$
$$c_2 = ip \cdot \sin\alpha.$$

Si noti che

$$\frac{c_2}{c_1} = \frac{\sin\alpha}{\cos\alpha}$$

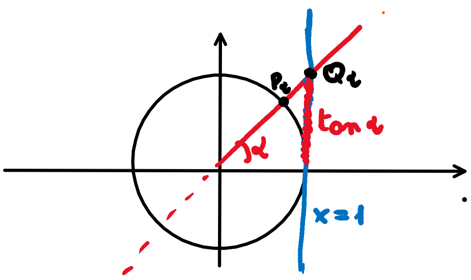
TANGENTE DI α

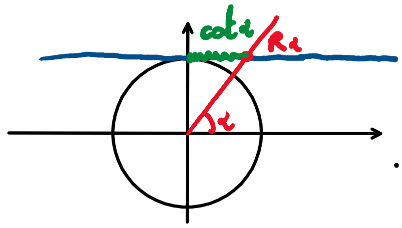
($\tan\alpha$, $tg\,\alpha$)

$$\frac{c_1}{c_2} = \frac{\cos\alpha}{\sin\alpha}$$

COTANGENTE DI α

($\cot\alpha$, $cotg\,\alpha$, $cotan\,\alpha$)





Valori di tan e cot

α	$\cos\alpha$	$\operatorname{sen}\alpha$	$\tan\alpha$	$\cot\alpha$
0	1	0	0	n.d
$\frac{\pi}{2}$	0	1	n.d	0
π	-1	0	0	n.d
$\frac{3}{2}\pi$	0	-1	n.d	0
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1
$\frac{3}{4}\pi$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	-1
$\frac{5}{4}\pi$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1	1
$\frac{7}{4}\pi$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	-1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
$\frac{2}{3}\pi$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$
$\frac{5}{6}\pi$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$
$-\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	-1

Formule utili

FORMULE DI ADDIZIONE E SOTTRAZIONE

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

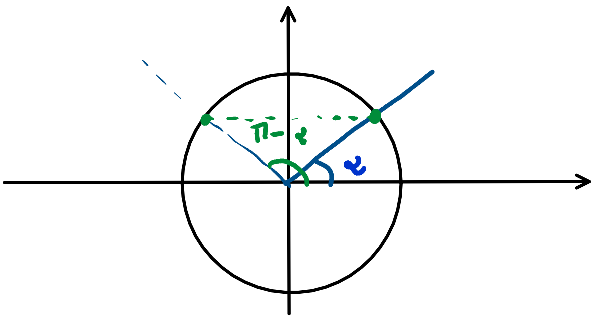
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Casi particolari:

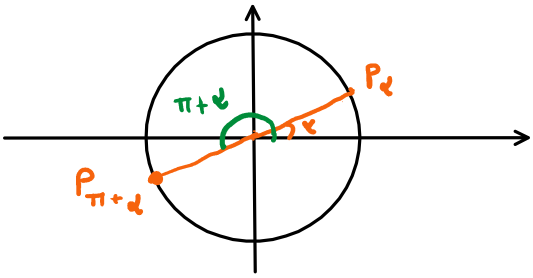
$$\cos(\pi-\alpha) = -\cos\alpha$$

$$\sin(\pi-\alpha) = \sin\alpha$$



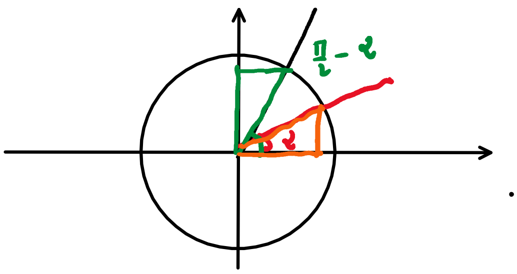
$$\cos(\pi+\alpha) = -\cos\alpha$$

$$\sin(\pi+\alpha) = -\sin\alpha$$



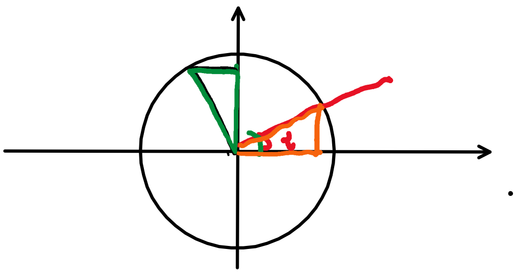
$$\cos\left(\frac{\pi}{2}-\alpha\right) = \sin\alpha$$

$$\sin\left(\frac{\pi}{2}-\alpha\right) = \cos\alpha$$



$$\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$$

$$\sin\left(\frac{\pi}{2}+\alpha\right) = \cos\alpha$$



2) FORMULE DI DUPLICAZIONE

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

3) FORMULE DI BISEZIONE

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} \qquad \sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$$

$\left(\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}\right.$, $\operatorname{sen}\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$

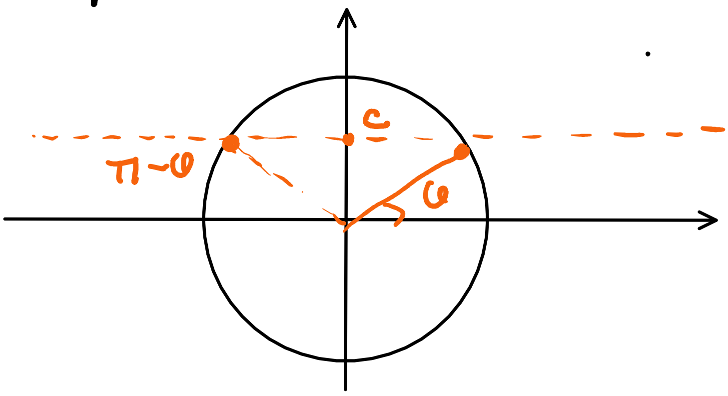
con il segno da determinare in base ad α)

ESEMPI

$$\cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{1+\cos\left(\frac{\pi}{4}\right)}{2}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}}$$

$$\cos\left(\frac{9}{8}\pi\right) = -\sqrt{\frac{1+\cos\left(\frac{9}{4}\pi\right)}{2}} = -\sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}}$$

Come si risolve un'equazione del tipo $\sin x = c$ con $c \in \mathbb{R}$?



$\sin x = c$

- Se $c > 1$ no soluzioni
- Se $c = 1$: $x = \frac{\pi}{2} + 2k\pi$ con $k \in \mathbb{Z}$
- Se $-1 < c < 1$:

  Se α è un qualsiasi angolo tale che $c = \sin \alpha$

  $x = \alpha + 2k\pi$ con $k \in \mathbb{Z}$ $\vee$ $x = \pi - \alpha + 2k\pi$
- $c = -1$: $x = \frac{3}{2}\pi + 2k\pi$ con $k \in \mathbb{Z}$
- $c < -1$: no soluzioni

ESEMPIO

$2 \sin x = 1$

$\sin x = \frac{1}{2}$

$\sin x = \sin \frac{\pi}{6}$

$x = \frac{\pi}{6} + 2k\pi \quad \vee \quad x = \pi - \frac{\pi}{6} + 2k\pi$

$x = \frac{\pi}{6} + 2k\pi \quad \vee \quad x = \frac{5}{6}\pi + 2k\pi$