On some generalizations of the implicit Euler method for discontinuous fractional differential equations[☆]

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Abstract

We discuss the numerical solution of differential equations of fractional order with discontinuous right–hand side. Problems of this kind arise, for instance, in sliding mode control. After applying a set–valued regularization, the behavior of some generalizations of the implicit Euler method is investigated. We show that the scheme in the family of fractional Adams methods possesses the same chattering–free property of the implicit Euler method in the integer case. A test problem is considered to discuss in details some implementation issues and numerical experiments are presented.

Keywords: fractional differential equation, discontinuous problem, set–valued regularization, implicit method, fractional Adams method.
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1. Introduction

The term fractional calculus refers to the generalization of integration and differentiation to any arbitrary (i.e., non necessarily integer) order. This idea is by no means new and was pioneered by Leibniz who mentioned the possibility of derivatives of order 1/2 in a correspondence exchanged with L'Hospital in 1695; successively, fractional calculus stimulated many famous mathematicians, including Euler, Fourier, Lagrange, Laplace, Riemann and some others.

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Despite this long history, relevant applications of fractional calculus have emerged only a few decades ago; nonetheless, models of fractional order are nowadays commonly used in several areas, ranging from chemistry and physics to biology, engineering, finance and so on. See [34] for an historical survey of fractional calculus.

In control theory it has been observed that the introduction of controllers involving integration or derivation of non–integer order allows to achieve higher performance with respect to classical systems of integer order; we refer the reader to one of the recent monographes [6, 33, 35, 37] on fractional systems.

Sliding mode control (SMC) is a special class of variable-structure systems; it is designed to alter the dynamics of the system which is firstly driven toward a switching surface and hence is constrained to stay on it. The success and widespread use of SMC is mainly due to its simplicity and robustness against parameter variations and disturbances [42].

Very recently the investigation of the effects of SMC techniques on fractional systems has been approached (e.g, see [4, 18, 36, 38, 39]). Since in SMC the control law is not a continuous function but switches from one continuous structure to another (according to the position of the system in the portion of the state delimited by the switching surface), its use for fractional order systems poses new and significant challenges, especially for the numerical computation.

In [2] the behavior of the implicit Euler (IE) method for the numerical simulation of non–smooth dynamical systems of integer order has been analyzed. It has been showed that, unlike the explicit Euler method which generates unwanted spurious oscillations, the implicit scheme allows a smooth stabilization on the switching surface. This important feature, which has been achieved after recasting the system into a Filippov's differential inclusion framework, validates the IE as a viable method for chattering suppression.

The main aim of this paper is to introduce the study of the counterparts of the IE method for fractional differential equations (FDEs) when applied to problems with discontinuous right–hand side. As it is known, the IE method can be generalized to FDEs according to different approaches, which give rise to different methods: we intend to verify whether implicit schemes are able to prevent chattering phenomena also in the fractional case and detect which of the generalizations of the IE method possess this feature. Furthermore we intend to discuss some of the major issues for the implementation of implicit methods in the context under investigation.

This paper is organized as follows. In Section 2 some basic facts on fractional calculus are reviewed, FDEs with discontinuity are introduced and some results concerning the Filippov's regularization of discontinuous FDEs are discussed. In Section 3 we present some of the most used numerical methods for FDEs. Interestingly, their application to a discontinuous test FDE in Section 4 discloses some unexpected features: different schemes leading to the same method when the order α of the FDE tends to the nearest integer (i.e., when the FDE tends to an ordinary differential equation (ODE)) behave in a different way in the presence of discontinuities; furthermore, only the method belonging to the class of fractional Adams methods seems to preserve the chattering–free motion observed for the IE method in the integer case. In Section 5 the attention is moved to the more general test problem introduced in [39] and we discuss some practical aspects related to the use of implicit methods. Finally, in Section 6 we present the results of some numerical simulations.

2. Differential inclusions of fractional order

Historically, the origins of fractional calculus are strictly related to the *Riemann–Liouville* definition of the integral of order $\alpha > 0$ on the interval $[t_0, t]$

$$J_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \big(t-s\big)^{\alpha-1} f(s) ds,$$

where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the Euler gamma function (for references to introductory material on fractional calculus the reader is referred to any classical textbook on the subject, for instance [15, 28, 31, 40]).

The definition of differential operators of fractional order is not unique and different approaches have been proposed. For instance, the *Riemann–Liouville* (RL) differential operator of order α is defined as

$${}^{RL}D_{t_0}^{\alpha} f(t) \equiv D^m J_{t_0}^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t_0}^{t} \big(t-s\big)^{m-\alpha-1} f(s) ds,$$

where $m = \lceil \alpha \rceil$ is the smallest integer such that $m > \alpha$ and D^m and d^m/dt^m denote the standard derivative of integer order.

An alternative definition, commonly named as the *Caputo* differential operator, has been introduced in [7, 8] and it is defined according to

$${}^{C}D_{t_0}^{\alpha} f(t) \equiv J_{t_0}^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \big(t-s\big)^{m-\alpha-1} f^{(m)}(s) ds.$$

Although the two definitions are very similar, we outline that some important differences follow from the interchange of $J_{t_0}^{m-\alpha}$ and D^m. It is beyond the scope of this paper to discuss in depth these theoretical aspects; we just highlight that, under appropriate assumptions of smoothness for f, the two definitions are strictly related by the relationship

$$ {}^{C}D_{t_0}^{\alpha} f(t) = {}^{RL}D_{t_0}^{\alpha} \big(f(t) - T_{m-1}[f;t_0](t) \big), $$

where $T_{m-1}[f;t_0]$ denotes the $(m-1)$-th degree Taylor polynomial for f centered at t_0

$$ T_{m-1}[f;t_0](t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(t_0)}{k!}(t-t_0)^k. $$

Thus, under homogeneous initial conditions $f(t_0) = f'(t_0) = \cdots = f^{(m-1)}(t_0) = 0$ the two approaches coincide. Furthermore it is immediate to verify that ${}^{RL}D_{t_0}^{\alpha}$ and ${}^{C}D_{t_0}^{\alpha}$ coincide with D^m when $\alpha \to m$.

A noticeable property is that fractional derivatives at a point t depend on the values assumed by the function over the entire interval $[t_0, t]$; this *non–locality* of fractional operators is one of the main properties which makes them suitable for modeling systems with memory effects.

For simplicity, in the discussion to follow we will assume $t_0 = 0$ and, in a general way, we will indicate with D_0^{α} a differential operator of fractional order α, with respect to the origin, according to one of the above definitions.

A system of differential equations of fractional order $\alpha > 0$ can be stated as

$$ D_0^{\alpha} y(t) = f(y(t)), \tag{1} $$

where $f : \mathcal{G} \to \mathbb{R}^q$, with $\mathcal{G} \subset \mathbb{R}^q$ a suitable domain. According to the approach used to define the fractional differential operator, a different set of initial conditions are coupled to (1). When D_0^{α} denotes the RL operator, the initial conditions are of the form

$$ D_0^{\alpha-k} y(0) = b_k, \quad k = 1, \ldots, m-1, \quad \lim_{t \to 0^+} J_0^{m-\alpha} y(t) = b_m. $$

The lack of a clear physical interpretation of initial data b_k is one of the main reasons which drove to the introduction of the Caputo's alternative definition which allows to couple the FDE with classical initial conditions of Cauchy type

$$ D^k y(0) = y_{0,k}, \quad k = 0, 1, \ldots, m-1. $$

A common characteristic of both approaches is that the true solution of (1) can be written in term of a Volterra integral equation of the second kind

$$ y(t) = Y_{\alpha}(t) + J_0^{\alpha} f(y(t)), \tag{2} $$

where the known term $Y_\alpha(t)$ depends on the initial conditions and on the type of operator and is given by

$$Y_\alpha(t) = \begin{cases} \displaystyle\sum_{k=1}^{m} \frac{t^{\alpha-k}}{\Gamma(\alpha-k+1)} b_k & \text{for } D_0^\alpha \text{ the RL operator;} \\ \displaystyle T_{m-1}[y;0](t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} y_{0,k} & \text{for } D_0^\alpha \text{ the Caputo operator.} \end{cases}$$

As a consequence, due to the presence of the negative power $t^{\alpha-m}$, the solution according to the RL approach is unbounded as $t \to 0^+$ (unless $b_m = 0$), whereas with the Caputo definition the true solution can be expected to be continuous at the origin but its successive derivatives present some singularities (see also [29]).

The continuity of f is usually sufficient to ensure the existence of the solution on a proper domain; furthermore, when f fulfills a Lipschitz condition the uniqueness of the solution is also achieved (we refer to [15] for a comprehensive list of results on existence and uniqueness of solutions of FDEs).

In several applications, especially in the context of control theory as discussed in the Introduction, the function f presents some discontinuities; when the vector field possesses only a piecewise smooth character the existence of a solution is no longer guaranteed.

To overcome most of the difficulties related to ODEs with a non smooth character, Filippov [19] proposed the following regularization: denoted with $\mathcal{B}(\mathbb{R}^q)$ the set of all subsets of $\mathbb{R}^q$, the vector field $f : \mathbb{R}^q \to \mathbb{R}^q$ is replaced by the set–valued map $F : \mathbb{R}^q \to \mathcal{B}(\mathbb{R}^q)$ obtained by evaluating f in the open ball $B(y, \delta)$ centered at y with radius $\delta > 0$ and for smaller and smaller δ; to obtain the same map F for vector fields which differ by a set of measure zero, the regularization excludes any set of measure zero from $B(y, \delta)$. From a more rigorous mathematical point of view the set–valued map F can be written as

$$F(y) = \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \overline{co}\big\{ f\big(B(y,\delta)\backslash S\big)\big\}, \tag{3}$$

where μ is the Lebesgue measure and $\overline{co}$ denotes the convex closure. The above process of regularization usually goes under the name of *Filippov's convexification*. For a recent discussion of theoretical and computational aspects on Filippov differential systems we refer, for instance, to [14] and references therein.

When $f(y)$ is locally bounded its corresponding set–valued map $F(y)$ is upper semicontinuous, i.e. for any y in the domain of F and for any open set Z such that $F(y) \subset Z$ there exists a neighborhood Y of y such that $F(Y) \subset Z$; furthermore $F(y)$ assumes nonempty convex values and $F(y) = \{f(y)\}$ whenever f is continuous at y (see [3, Chapter 2, §1, Prop. 1]).

The above process of regularization can be extended to discontinuous FDEs. Thus, a Filippov solution for problem (1) is any absolutely continuous map $y(t) : [0, T] \to \mathbb{R}^q$ such that the following differential inclusion

$$D_0^\alpha y(t) \in F(y) \tag{4}$$

is satisfied for almost all $t \in [0, T]$, with F the Filippov's convexification of f according to (3).

The existence of solutions for differential inclusions of fractional order is a topic which has begun to be discussed only recently (for instance, see [9, 10, 12]). We provide an existence result for (4) by operating within the framework established in [12].

Proposition 2.1. *Let $\alpha > 0$ and assume f locally bounded. For each set of initial conditions, there exists $T > 0$ and an absolutely continuous function $y(t)$ on $[0, T]$ such that (4) holds for almost $t \in [0, T]$.*

Proof. Since the vector field of (4) is upper semicontinuous, the Approximate Selection Theorem [3] ensures that for any $\varepsilon > 0$ there exists a locally Lipschitzean map f_ε such that

$$Graph(f_\varepsilon) \subset Graph(F) + \varepsilon B,$$

where B is the unitary ball centered at the origin. Thus the standard FDE $D_0^\alpha y(t) = f_\epsilon(y)$ admits a solution and the proof follows by applying the Convergence Theorem in [3]. □

3. Numerical methods for FDEs

Recently, there has been a spread in the development and analysis of numerical methods for FDEs. In the following we will briefly recall some of the main approaches followed to develop numerical methods for the time discretization of FDEs; the approaches we are going to present are not only some of the most frequently mentioned in literature but are also interesting since they generalize well–known methods for ODEs.

3.1. Product integration rules

Product integration (PI) is a well established numerical technique [43] in which, given a mesh–grid $t_n = nh$ on the interval of integration $[0, T]$, the integrand function $f(y(s))$ in the VIE (2) is replaced, in each subinterval $[t_j, t_{j+1}]$, by a suitable piecewise interpolant polynomial of a certain degree k. Convergence of PI rules was first studied in a general way by Cameron and McKee [5] who found that, under smoothness assumptions on the true solution of (2), the resulting method converges with order $p = k + 1$ as the step–size $h \to 0$. Anyway, since the derivatives of the true solution of (2) usually present a singularity at the origin, in general situations the actual order of PI rules does not exceed $1 + \alpha$ as successively proved in [17].

Splitting the integration interval into the subintervals defined by the grid points leads to the expression

$$J_0^\alpha f(y(t_n)) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \bigl(t_n - s\bigr)^{\alpha-1} f(y(s)) ds;$$

when the function $f(y(s))$ is replaced in each subinterval $[t_j, t_{j+1}]$ by the constant value $f(y_{j+1})$ the resulting method is the implicit PI rectangular rule

$$\begin{cases} Y_n = Y_\alpha(t_n) + h^\alpha \displaystyle\sum_{j=1}^{n-1} b_{n-j} f(y_j) \\ y_n - Y_n = h^\alpha b_0 f(y_n) \end{cases}, \quad (5)$$

where $b_n = \left((n+1)^\alpha - n^\alpha\right)/\Gamma(\alpha+1)$ and y_n is the approximation of the solution $y(t)$ at the grid–point t_n.

An explicit counterpart of (5) is easily obtained, in a similar way, by using in each subinterval $[t_j, t_{j+1}]$ the constant value $f(y_j)$ instead of $f(y_{j+1})$. It is straightforward that the resulting explicit PI rectangular rule is

$$y_n = Y_\alpha(t_n) + h^\alpha \sum_{j=0}^{n-1} b_{n-j-1} f(y_j) \quad (6)$$

(the reason for which methods (5) and (6) are written in a slightly different way will be clear later). Both methods converge with order 1 as $h \to 0$. It is easy to verify that as $\alpha \to 1$ (hence when the FDE tends to an ODE) the above methods reduce respectively to the classical (implicit Euler) IE and explicit Euler (EE) methods.

When linear interpolants are used for replacing the function f, the PI trapezoidal rule is obtained. This is mainly used in the context of a predictor–evaluate–corrector–evaluate (PECE) framework for fractional Adams-Bashforth-Moulton (ABM) methods [16] in which a preliminary approximation evaluated by the explicit method (6) is successively refined by means of one or more iterations of the implicit PI trapezoidal rule. Although based on an implicit method, the PECE method is applied in an explicit way and inherits advantages (such as low computational cost) and weaknesses (such as limitations in the stability region [24]) of explicit methods. The diffusion of this method among the scientific community is remarkable; it has been also proposed for the approximation of discontinuous fractional systems in [13].

More recently, PI rules have been employed in [25, 26] to develop fractional generalizations of exponential quadratures and integrators.

3.2. Fractional Adams methods

The family of fractional Adams–Moulton (FAM) methods has been studied in [21] with the aim of simplifying the evaluation of the coefficients of fractional Newton–Gregory formulas previously introduced in [30]. The general formulation of a k–step FAM method, which has order of convergence $p = k + 1$, is

$$\sum_{j=0}^{n} \omega_j^{(\alpha)} \bigl(y_{n-j} - y_0\bigr) + \sum_{j=0}^{s} w_{n,j} \bigl(y_j - y_0\bigr) = h^\alpha \sum_{j=0}^{k} \gamma_j^{(k)} f(y_{n-j}) \tag{7}$$

where coefficients γ_j^k depend on α (they have been tabulated in [21] for $k = 0, \dots, 5$); $w_{n,j}$ are starting weights evaluated in order to deal with the behavior of the true solution near the origin and are chosen by imposing (7) to be exact for $y(t) = t^{j+\alpha i}$ for i and j integers and $j + \alpha i \le p - 1$. The weights $\omega_j^{(\alpha)}$ are the coefficients in the power series expansion of $(1 - \xi)^\alpha$, i.e.

$$(1-\xi)^\alpha = \sum_{j=0}^{\infty} \omega_j^{(\alpha)} \xi^j, \quad \omega_j^{(\alpha)} = \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}, \tag{8}$$

and can be evaluated by means of the recursive recurrence [34]

$$\omega_0^{(\alpha)} = 1, \quad \omega_j^{(\alpha)} = \left(1 - \frac{\alpha+1}{j}\right) \omega_{j-1}, \quad j = 1, 2, \dots. \tag{9}$$

The most simple method in this class is for $k = 0$; in this case starting weights are identically equal to 0 and, since it is [34, Equation (1.3.18)]

$$\sum_{j=0}^{n} \omega_j^{(\alpha)} = \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)} = \omega_n^{(\alpha-1)}, \tag{10}$$

the scheme can be reformulated as

$$\begin{cases} Y_n = \omega_n^{(\alpha-1)} y_0 - \displaystyle\sum_{j=0}^{n-1} \omega_{n-j}^{(\alpha)} y_j \\ y_n - Y_n = h^\alpha f(y_n) \end{cases} . \tag{11}$$

Although, in general, FAM methods are different from the fractional BDF methods introduced in [30], the special case of method (11) is equivalent to the fractional BDF rule of order 1 and, moreover, it corresponds to the application of the Grünwald-Letnikov discretization to $y(t) - y_0$ [41].

An explicit version of this method has been studied firstly in [20] and successively, in a more general way, in [23] under the name of fractional Adams Bashforth (FAB) methods; it is given by

$$y_n - Y_n = h^\alpha f(y_{n-1}), \quad n \geq 1, \tag{12}$$

where the lag–term Y_n is evaluated in the identical manner as in (11).

When $\alpha = 1$, it is easy to see that $\omega_1^{(\alpha)} = -1$, $\omega_n^{(\alpha)} = 0$ for $n \geq 2$ and $\omega_n^{(\alpha-1)} = 0$ for $n \geq 1$. Thus $Y_n = y_{n-1}$ and (11) is the above methods become respectively the IE and the EE method as it happens with the PI rules.

Coefficients $\omega_n^{(\alpha)}$ reveal some important features. We present here two results which will be used in Section 4.

Lemma 3.1. *Let $0 < \alpha < 1$ and $\omega_n^{(\alpha)}$ the coefficients in the power series expansion of $(1-\xi)^\alpha$ given in (8). Then for any $n = 1, 2, \ldots$*

a) $-1 < \omega_n^{(\alpha)} < 0$*;*

b) $0 < \omega_n^{(-\alpha)} < 1$*;*

c) $\omega_{n-1}^{(-\alpha-1)} > \omega_n^{(-\alpha-1)} - 1$.

Proof. Points *a)* and *b)* are immediate consequences of the recursive relationship (9). For point *c)* observe that

$$\omega_n^{(-\alpha-1)} = \left(1 + \frac{\alpha}{n}\right) \omega_{n-1}^{(-\alpha-1)} = \omega_{n-1}^{(-\alpha-1)} + \frac{\alpha}{n} \omega_{n-1}^{(-\alpha-1)}$$

and moreover

$$\frac{\alpha}{n} \omega_{n-1}^{(-\alpha-1)} = \frac{\alpha \Gamma(n+\alpha)}{n \Gamma(\alpha+1) \Gamma(n)} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} = \omega_n^{(-\alpha)}.$$

Hence $\omega_n^{(-\alpha-1)} = \omega_{n-1}^{(-\alpha-1)} + \omega_n^{(-\alpha)}$ and thanks to point *b)* it is $\omega_n^{(-\alpha-1)} < \omega_{n-1}^{(-\alpha-1)} + 1$ from which the proof follows. □

Lemma 3.2. *Let $0 < \alpha < 1$ and $\omega_n^{(\alpha)}$ the coefficients in the expansion of $(1-\xi)^\alpha$ in (8). For any $M \in \mathbb{R}$ the difference equation*

$$Z_n + \sum_{j=0}^{n-1} \omega_{n-j}^{(\alpha)} Z_j = M \sum_{j=1}^{n} \omega_j^{(\alpha)}, \quad n \geq 1, \tag{13}$$

with $Z_0 = 0$, is solved by $Z_n = M\left(1 - \omega_n^{(-\alpha-1)}\right)$.

Proof. We observe as a preliminary that, since $\omega_0^{(\alpha)} = 1$ and thanks to (10), the difference equation (13) can be equivalently written as

$$\sum_{j=0}^{n} \omega_{n-j}^{(\alpha)} Z_j = M\left(\omega_j^{(\alpha-1)} - 1\right).$$

Denote with $Z(\xi)$ the formal power series $Z(\xi) = \sum_{k=0}^{\infty} Z_k \xi^k$ corresponding to the sequence $\left\{Z_n\right\}_{n\in\mathbb{N}}$. Since the formal power series corresponding to the constant sequence of constant value 1 and to the sequence $\left\{\omega_n^{(\alpha)}\right\}_{n\in\mathbb{N}}$ are $(1-\xi)^{-1}$ and $(1-\xi)^\alpha$ respectively, by the usual Cauchy product of formal power series it is immediate to show that

$$Z(\xi)(1-\xi)^\alpha = M\left((1-\xi)^{\alpha-1} - \frac{1}{1-\xi}\right)$$

from which the proof can be easily concluded after writing

$$Z(\xi) = M\left(\frac{1}{1-\xi} - \left(1-\xi\right)^{-\alpha-1}\right).$$

□

4. Numerical treatment of discontinuous FDEs

To provide a preliminary analysis of the behavior of the numerical schemes described in Section 3 when employed for the time simulation of fractional discontinuous systems, we first consider the simple test problem

$$\begin{cases} D_0^\alpha y(t) = -\lambda \,\mathrm{sgn}(y(t)) \\ y(0) = y_0 \end{cases} \tag{14}$$

where $0 < \alpha < 1$, $\lambda > 0$ and $\mathrm{sgn}(x)$ is the classical signum function

$$\mathrm{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ +1 & \text{if } x > 0 \end{cases},$$

whose graph is drawn in the first plot of Figure 1. The switching surface $\sigma(y) = 0$ for this simple test problem is defined by $y = 0$, being $\sigma(y) = y$.

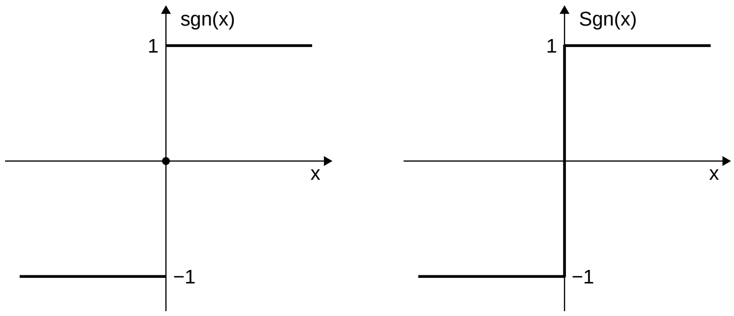

Figure 1: Graph of the functions $\mathrm{sgn}(x)$ and $\mathrm{Sgn}(x)$.

By direct integration, we can readily observe that, for $y_0 \neq 0$, the dynamics of the solution $y(t)$ of (14) initially starts as

$$y(t) = y_0 - \lambda\,\mathrm{sgn}(y_0)\frac{t^\alpha}{\Gamma(\alpha+1)};$$

this *reaching phase* drives the system state toward the switching surfaces $\sigma(y) = 0$ which is reached at the time instant

$$t_s = \left(\frac{\Gamma(\alpha+1)}{\lambda}|y_0|\right)^{\frac{1}{\alpha}}.$$

The stability of the reaching dynamics of fractional order systems has been discussed in [18], where the globally attracting character of the switching surface $\sigma(y) = 0$ under the action of (14) has been demonstrated. Indeed it has been showed that a control law ensuring $yy^\alpha < 0$ also ensures the standard reachability condition $yy' < 0$ of integer order systems, thus providing stability to the fractional system (14).

In Figure 2 we show the motion of the true solution of (14) for two selections of the main parameters, respectively $\alpha = 0.35$, $\lambda = 0.85$ and $\alpha = 0.65$, $\lambda = 0.55$ and for the initial conditions $y_0 = \pm 1$.

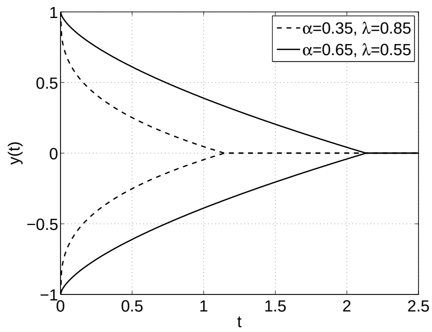

Figure 2: System state of (14) for $y_0 = 1$ (upper part) and $y_0 = -1$ (lower part)

Numerical methods of explicit type can be directly applied to discretize problems like (14). As for the case of ODEs, studied in detail in [22], explicit methods introduce spurious oscillations as shown in Figure 3 where the explicit PI rule (6) and the explicit FAB method (12) have been used for $\alpha = 0.35$ and $\lambda = 0.85$ (the solid line denotes the state y_n and the dotted line the control $u_n = -\lambda \operatorname{sgn}(y_n)$). A step–size $h = 2^{-4}$ has been used in all the simulations; a smaller step–size does not eliminate this chattering effect but just reduces the amplitude of the undesired oscillations whose frequency, at the same time, increases.

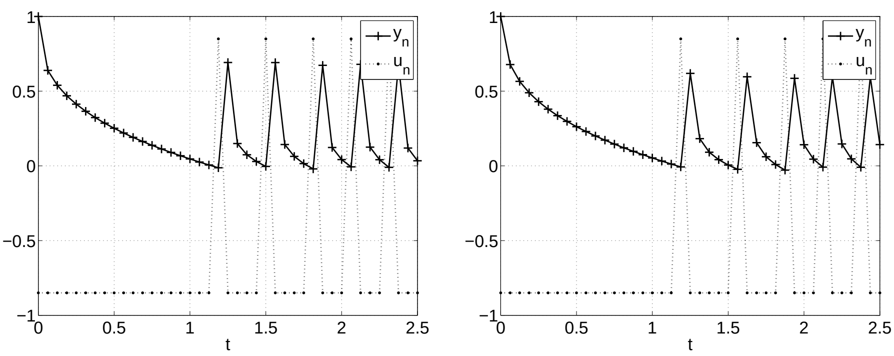

Figure 3: Explicit PI rule (6) (left) and FAB method (12) (right) for $\alpha = 0.35$, $\lambda = 0.85$ and $y_0 = 1$.

The same kind of spurious oscillations appear also in some experiments (not reported here for brevity) performed with the PECE method which, as previously observed, can be considered, from a computational point of view, an explicit method.

Implicit methods can not be applied directly to problem (14). Indeed, the presence of a discontinuity on the vector field near the switching surface does not allow to solve the equation deriving from the implicitness of the method. Thus, the regularization of the problem in the framework established by Filippov is necessary also for applying numerical methods of implicit type. For the problem under investigation this means to replace (14) with the differential inclusion

$$\begin{cases} D_0^{\alpha} y(t) \in -\lambda \operatorname{Sgn}(y(t)) \\ y(0) = y_0 \end{cases} \tag{15}$$

where Sgn denotes the set–valued function obtained by applying the Filippov's convexification (3) to the standard sgn function. The function $\operatorname{Sgn}(x)$ is shown in the second plot of Figure 1 and is defined as

$$\operatorname{Sgn}(x) = \begin{cases} \{-1\} & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ \{+1\} & \text{if } x > 0 \end{cases} .$$

When applied to problem (15) implicit methods (5) and (11) can be written as

$$\begin{cases} y_n = \bar{Y}_n - h^{\alpha} \bar{a} \lambda s_n \\ s_n \in \operatorname{Sgn}(y_n) \end{cases}$$

where $\bar{Y}_n$ is the proper lag–term and $\bar{a}$ the coefficient $\bar{a} = 1/\Gamma(\alpha+1)$ for (5) and $\bar{a} = 1$ for (11). Solving this simple test problem consists in evaluating the intersection between the graph of the multivalued mapping $y_n \to -h^{\alpha}\bar{a}\lambda \operatorname{Sgn}(y_n)$ and the straight line $y_n \to y_n - \bar{Y}_n$ [2].

As we can see in Figure 4 the implicit PI rule (5) continues to present spurious oscillations, even if of reduced amplitude with respect to the explicit PI rule, while the implicit FAM method (11) assures a smooth stabilization along the sliding surface.

We have verified that oscillations arise also with the more accurate implicit PI trapezoidal rule, thus confirming that the appearance of oscillations is a feature of the family of PI rules and not an issue depending on the accuracy of the method.

The robustness of the FAM method (11) does not depend on the step–size h. We can provide the following result, playing for FDEs the analogue role of Lemma 1 in [2] for ODEs, which shows that the FAM method (7) assures a smooth stabilization on the sliding surface for any step–size h.

Proposition 4.1. *Let $0 < \alpha < 1$ and $\{y_n\}_{n\in\mathbb{N}}$ the solution evaluated by the FAM method (11) with step–size h. For any $h > 0$ and $y_0 \in \mathbb{R}$, there exists $\bar{n} \in \mathbb{N}$ such that $y_{\bar{n}+k} = 0$ for any $k \geq 1$.*

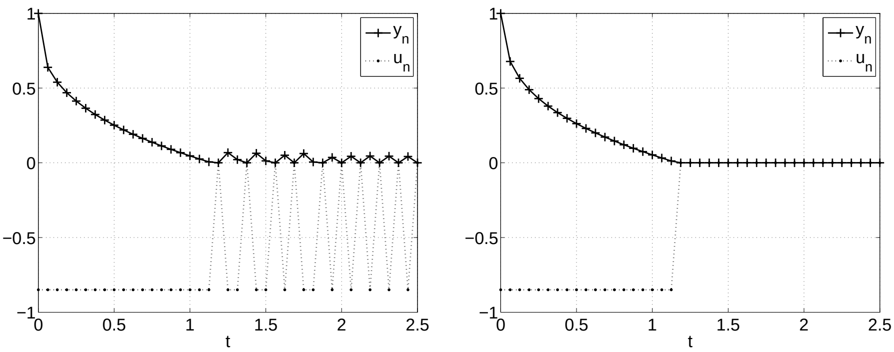

Figure 4: Implicit PI rule (5) (left) and FAM method (11) (right) for $\alpha = 0.35$ and $\lambda = 0.85$.

Proof. We prove this result for $y_0 > 0$ (when $y_0 = 0$ the proof is trivial and for $y_0 < 0$ we can proceed in a symmetric way). We first assume that $y_0 > h^\alpha \lambda$. Since $Y_1 = y_0$, it will be $y_n = Y_n - h^\alpha \lambda$ as long as $Y_n > h^\alpha \lambda$. Denote with $\bar{n}$ the first time–step such that $Y_{\bar{n}} \in [0, h^\alpha \lambda]$. After putting $Z_n = Y_n - y_0$, from (7) we have

$$Z_n = -\sum_{j=0}^{n-1} \omega_{n-j}^{(\alpha)} Z_j + h^\alpha \lambda \sum_{j=1}^{n} \omega_j^{(\alpha)}$$

for $n = 1, \dots, \bar{n}$ and, since $Z_0 = 0$, the application of Lemma 3.2 leads to

$$Y_n = y_0 + h^\alpha \lambda \left(1 - \omega_n^{(-\alpha-1)}\right), \quad n = 0, \dots, \bar{n}. \tag{16}$$

Therefore $\bar{n}$ is the smallest integer such that $\omega_{\bar{n}}^{(-\alpha-1)} \geq y_0/(h^\alpha \lambda)$. In the case $y_0 \in (0, h^\alpha \lambda]$ it trivially follows $\bar{n} = 0$.

Since $\omega_{\bar{n}}^{(-\alpha-1)} - 1 < \omega_{\bar{n}-1}^{(-\alpha-1)}$ (see Lemma 3.1) and $\omega_{\bar{n}-1}^{(-\alpha-1)} < y_0/h^\alpha$ it is immediate to verify that $Y_{\bar{n}} > 0$.

We now proceed by induction. Let k be an integer such that $Y_{\bar{n}+j} \in (0, h^\alpha \lambda]$ for $j = 0, 1, \dots, k-1$ (obviously k is at least 1 since $Y_{\bar{n}} \in (0, h^\alpha \lambda]$) and we prove that $Y_{\bar{n}+k} \in (0, h^\alpha \lambda]$ too.

We first observe that, since $y_n = Y_n - h^\alpha \lambda$ for $n = 0, 1 \dots, \bar{n} - 1$ and $y_n = 0$ for $n = \bar{n}, \bar{n}+1, \dots, \bar{n}+k-1$, thanks to (16) it is $y_n = y_0 - h^\alpha \lambda \omega_n^{(-\alpha-1)}$ as $n = 0, 1 \dots, \bar{n}-1$ and hence for $n \in \{0, 1, \dots, \bar{n}+k-1\}$ the sequence of y_n is decreasing with $y_n \geq 0$. Thus, from (7) it immediately follows $Y_{\bar{n}+k} \geq 0$

too. Moreover we can write

$$Y_{\bar{n}+k} = \sum_{j=0}^{\bar{n}+k-1} \omega_j^{(\alpha)} y_0 - \sum_{j=1}^{\bar{n}-1} \omega_j^{(\alpha)} y_{\bar{n}+k-j} - \sum_{j=\bar{n}}^{\bar{n}+k-1} \omega_j^{(\alpha)} y_{\bar{n}+k-j}$$

and since $y_{\bar{n}+k-j} \le y_{\bar{n}-j}$ we have

$$Y_{\bar{n}+k} \le \sum_{j=0}^{\bar{n}-1} \omega_j^{(\alpha)} y_0 - \sum_{j=1}^{\bar{n}-1} \omega_j^{(\alpha)} y_{\bar{n}-j} + \sum_{j=\bar{n}}^{\bar{n}+k-1} \omega_j^{(\alpha)} y_0 - \sum_{j=\bar{n}}^{\bar{n}+k-1} \omega_j^{(\alpha)} y_{\bar{n}+k-j}$$

and, after adding and subtracting $\omega_{\bar{n}}^{(\alpha)} y_0$, we obtain

$$Y_{\bar{n}+k} \le Y_{\bar{n}} - \sum_{j=\bar{n}}^{\bar{n}+k-1} \omega_j^{(\alpha)} \big(y_{\bar{n}+k-j} - y_0 \big) \le Y_{\bar{n}} \le h^{\alpha} \lambda.$$

We have thus proved that $Y_{\bar{n}+k} \in (0, h^{\alpha}\lambda]$, from which the induction thesis $y_{\bar{n}+k} = 0$ follows in an easy way. □

As it can be clearly evinced from Figure 4 (left), a similar property of robustness can not be proved for the PI rule (5). We think that this result is particularly important, especially considering that the PI rules, and most of all the PECE method based on them, are some of the most frequently used methods in the time–simulation of fractional order systems. Thus, a particular attention should be payed when PI rules are employed for the time–discretization of discontinuous FDEs. For this reason, in the following our investigation will be confined to the FAM method (11).

5. A linear multivariable system of fractional order

In this Section we intend to discuss in greater detail the application of the implicit FAM method (11) to a more general test problem. We consider the linear multivariable system of fractional order analyzed in [39]

$$D_0^{\alpha} y(t) = A y(t) + B\big(u(t) + d(t)\big), \tag{17}$$

where $y(t) \in \mathbb{R}^q$ is the state–vector, $u(t) \in \mathbb{R}^m$ the control vector and $d(t) \in \mathbb{R}^m$ an uncertain disturbance for which it is assumed the existence of a known bound $d_M(t)$ such that $\|d(t)\| \le d_M(t)$. $A \in \mathbb{R}^{q\times q}$ and $B \in \mathbb{R}^{q\times m}$ denote the characteristic and control matrices respectively; in order to assure that the system can reach any point in the state space through the choice of an appropriate control input, according to the classical theory of control system the pair (A, B) is assumed controllable, i.e. the $q \times qm$ controllability matrix $(B,\ AB,\ A^2B,\ \ldots,\ A^{q-1}B)$ has full rank q; also B is assumed to have full rank $m \leq q$.

The control $u(t)$ is selected as

$$u(t) = -(CB)^{-1} \left(CAy(t) - r\sigma(t) + (K + \mu d_M(t)) \operatorname{sgn}(\sigma(t)) \right),$$

where $r > 0$, $K > 0$ and $\mu > \|CB\|$ are scalar parameters, in order to force the trajectory of the system to remain on the m–dimensional sliding surface

$$\sigma(t) = CJ_0^{1-\alpha} y(t) = 0,$$

with $C \in \mathbb{R}^{m \times q}$. A standard manipulation allows to show that $\sigma(t)$ can be equivalently characterized in terms of the integer order ODE

$$\frac{d}{dt}\sigma(t) = CAy(t) + CB\big(u(t) + d(t)\big). \tag{18}$$

The FDE (17) is discretized, on a grid–mesh $t_n = nh$, by the FAM method (11) and, for consistency, the integration of (18) is carried out by means of the classical IE method

$$\sigma_n = \sigma_{n-1} + h\big(CAy_n + CBu_n + CBd(t_n)\big). \tag{19}$$

Thus, after putting

$$Y_n = \omega_{n-1}^{(\alpha-1)} y_0 - \sum_{j=1}^{n-1} \omega_{n-j}^{(\alpha)} y_j \in \mathbb{R}^q,$$

we can write the resulting discretized system as

$$\left\{ \begin{array}{l} y_n = Y_n + h^\alpha A y_n + h^\alpha B u_n + h^\alpha B d(t_n) \\ u_n = -(CB)^{-1} CA y_n + r(CB)^{-1} \sigma_n - (K + \mu d_M(t_n))(CB)^{-1} \operatorname{sgn}(\sigma_n) \end{array} \right. .$$

A simple manipulation and the use of (19) allow us to write

$$P \begin{pmatrix} y_n \\ u_n \end{pmatrix} = \begin{pmatrix} Y_n + h^\alpha B d(t_n) \\ r(CB)^{-1} \left(\sigma_{n-1} + hCBd(t_n) \right) - b_n \end{pmatrix}, \tag{20}$$

where

$$P = \begin{pmatrix} \big(I_q - h^\alpha A\big) & -h^\alpha B \\ (1 - hr)(CB)^{-1} CA & (1 - hr) I_m \end{pmatrix}, \tag{21}$$

$b_n = (K + \mu d_M(t_n))(CB)^{-1} \operatorname{sgn}\big(\sigma_{n-1} + hCAy_n + hCBu_n + hCBd(t_n)\big)$ depends on y_n and u_n in a discontinuous way and I_q denotes the identity matrix of size $q \times q$. In the remaining of this paper we will repeatedly make use of the following result, whose trivial proof is omitted.

Lemma 5.1. *Let A be a real matrix, I the identity matrix of the same size of A and $\xi \in \mathbb{R}$ any values such that $I - \xi A$ is non singular. Then* $I + \xi (I - \xi A)^{-1} A = (I - \xi A)^{-1}$.

To study some properties of the coefficient matrix P in the system (20) we introduce the following result.

Proposition 5.2. *For $h > 0$ sufficiently small the matrix P is invertible and*

$$P^{-1} = \begin{pmatrix} M & h^\alpha (I_q - h^\alpha A)^{-1} BN \\ -(CB)^{-1}CAM & N \end{pmatrix},$$

where the matrices $M \in \mathbb{R}^{q\times q}$ and $N \in \mathbb{R}^{m\times m}$ depend on α, h and r and are given respectively by

$$M = \Big(I_q - h^\alpha \Big(I_q - B\big(CB\big)^{-1}C\Big) A\Big)^{-1}, \quad N = \frac{1}{1-hr}\Big(C\big(I - h^\alpha A\big)^{-1}B\Big)^{-1} CB.$$

Proof. Denote, for simplicity, with $P_{i,j}$, $i,j = 1,2$, the blocks of P as they appear in (21). When $h < \min\left\{r^{-1}, \rho(A)^{-\frac{1}{\alpha}}\right\}$, with $\rho(A)$ the spectral radius of A, $P_{1,1}$ and $P_{2,2}$ are non singular and by means of block Gaussian elimination we can evaluate

$$P = \begin{pmatrix} I & P_{1,2}P_{2,2}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} P_{1,1} - P_{1,2}P_{2,2}^{-1}P_{2,1} & 0 \\ P_{2,1} & P_{2,2} \end{pmatrix}.$$

Thus we can evaluate the determinant of P

$$\det(P) = (1-hr)^m \det\left(I_q - h^\alpha \left(I_q - B(CB)^{-1}C\right) A\right)$$

and hence the non singularity of P is ensured for

$$h < \min\left\{r^{-1}, \rho(A)^{-\frac{1}{\alpha}}, \rho\left((I_q - B(CB)^{-1}C)A\right)^{-\frac{1}{\alpha}}\right\}. \tag{22}$$

A simple and standard computation allows to evaluate

$$P^{-1} = \begin{pmatrix} \Big(P_{1,1} - P_{1,2}P_{2,2}^{-1}P_{2,1}\Big)^{-1} & -P_{1,1}^{-1}P_{1,2}\Big(P_{2,2} - P_{2,1}P_{1,1}^{-1}P_{1,2}\Big)^{-1} \\ -P_{2,2}^{-1}P_{2,1}\Big(P_{1,1} - P_{1,2}P_{2,2}^{-1}P_{2,1}\Big)^{-1} & \Big(P_{2,2} - P_{2,1}P_{1,1}^{-1}P_{1,2}\Big)^{-1} \end{pmatrix}.$$

It is immediate to note that

$$P_{1,1} - P_{1,2}P_{2,2}^{-1}P_{2,1} = I_q - h^\alpha \left(I_q - B(CB)^{-1}C\right) A$$

and, since

$$P_{2,2} - P_{2,1}P_{1,1}^{-1}P_{1,2} = (1-hr)(CB)^{-1}C\left(I_m + h^\alpha A\,(I - h^\alpha A)^{-1}\right)B,$$

the application of Lemma (5.1) leads to

$$P_{2,2} - P_{2,1}P_{1,1}^{-1}P_{1,2} = (1-hr)(CB)^{-1}C\,(I - h^\alpha A)^{-1}\,B.$$

Furthermore, by evaluating

$$-P_{1,1}^{-1}P_{1,2}\left(P_{2,2} - P_{2,1}P_{1,1}^{-1}P_{1,2}\right)^{-1} = \frac{h^\alpha}{1-hr}\,(I_q - h^\alpha A)^{-1}\,B\left(C\,(I - h^\alpha A)^{-1}\,B\right)^{-1}CB,$$

the proof can be easily concluded. □

From now on we will assume h to be sufficiently small to guarantee that P is invertible (the proof of the above result explicitly provides the bound on h for the non–singularity of P). By denoting

$$\bar{y}_n = \begin{pmatrix} y_n \\ u_n \end{pmatrix} \in \mathbb{R}^{q+m}, \quad \bar{Y}_n = P^{-1}\begin{pmatrix} Y_n + h^\alpha Bd(t_n) \\ +r(CB)^{-1}\,(\sigma_{n-1} + hCBd(t_n)) \end{pmatrix} \in \mathbb{R}^{q+m},$$

$$\bar{Z}_n = \begin{pmatrix} 0_q \\ \sigma_{n-1} + hCBd(t_n) \end{pmatrix} \in \mathbb{R}^{q+m},$$

$$D_n = (K + \mu d_M(t_n))P^{-1}\begin{pmatrix} 0_{q\times q} & 0_{q\times m} \\ 0_{m\times q} & -(CB)^{-1} \end{pmatrix} \in \mathbb{R}^{(q+m)\times(q+m)},$$

and

$$F = h\begin{pmatrix} 0_{q\times q} & 0_{q\times m} \\ CA & CB \end{pmatrix} \in \mathbb{R}^{(q+m)\times(q+m)},$$

and observing that

$$\sigma_{n-1} + hCAy_n + hCBu_n + hCBd(t_n) = \sigma_{n-1} + hCBd(t_n) + hC\Big(A, B\Big)\bar{y}_n,$$

we can apply the set–valued Filippov regularization discussed in Section 2 and reformulate the system (20) as

$$\left\{\begin{array}{l} \bar{y}_n \in \bar{Y}_n + D_n\,\mathrm{Sgn}(z_n) \\ z_n = \bar{Z}_n + F\bar{y}_n \end{array}\right., \tag{23}$$

where $z_n \in \mathbb{R}^{q+m}$ and $\mathrm{Sgn}(z_n)$ is the component–wise composition of the Sgn function introduced in Section 4.

Solving (23) requires to find the intersection between the graph of the single–valued function $\bar{y}_n \to \bar{y}_n - \bar{Y}_n$ and the graph of the multivalued function $\bar{y}_n \to \mathrm{Sgn}(\bar{Z}_n + F\bar{y}_n)$. To solve this problem we need to recast it as a linear complementarity problem (LCP) and solve it numerically [1].

To this purpose first denote with x_n the vector in $\mathbb{R}^{q+m}$ such that $D_n x_n = \bar{y}_n - \bar{Y}_n$ and hence consider the slack variables $z_n^{(L)}, z_n^{(R)} \in \mathbb{R}^{q+m}$ with $z_n = z_n^{(R)} - z_n^{(L)}$. It is [27]

$$x_n \in \mathrm{Sgn}(z_n) \quad \Longleftrightarrow \quad \exists z_n^{(L)}, z_n^{(R)} \text{ s.t.} \begin{cases} 0 \le z_n^{(L)} \perp (e + x_n) \ge 0 \\ 0 \le z_n^{(R)} \perp (e - x_n) \ge 0 \end{cases}$$

where e denotes the vector $e = (1, 1, \dots, 1)^T$ of the same size of x_n. Thus we can express (23) in terms of complementarities,

$$\begin{cases} z_n^{(R)} - z_n^{(L)} = \bar{Z}_n + F\bar{Y}_n + FD_n x_n \\ 0 \le z_n^{(L)} \perp (e + x_n) \ge 0 \\ 0 \le z_n^{(R)} \perp (e - x_n) \ge 0 \end{cases} .$$

Consider now

$$v_n^{(L)} = \frac{1}{2}\big(e + x_n\big) \quad , \quad w_n^{(R)} = \frac{1}{2}\big(e - x_n\big) \tag{24}$$

and observe that

$$0 \le v_n^{(L)} \perp z_n^{(L)} \ge 0 \quad , \quad 0 \le w_n^{(R)} \perp z_n^{(R)} \ge 0$$

from which, by writing $v_n^{(R)} = z_n^{(R)}$ and $w_n^{(L)} = z_n^{(L)}$, it is easy to observe that

$$\begin{cases} z_n = v_n^{(R)} - w_n^{(L)} \\ 0 \le v_n^{(L)} \perp w_n^{(L)} \ge 0 \\ 0 \le v_n^{(R)} \perp w_n^{(R)} \ge 0 \end{cases} .$$

Furthermore from (24) we have

$$x_n = 2v_n^{(L)} - e \quad , \quad w_n^{(R)} = \frac{1}{2}\big(e - x_n\big) = \frac{1}{2}\big(e - 2v_n^{(L)} + e\big) = e - v_n^{(L)}$$

and hence

$$\begin{cases} v_n^{(R)} - w_n^{(L)} = \bar{Z}_n + F\bar{Y}_n + FD_n\big(2v_n^{(L)} - e\big) \\ w_n^{(R)} = e - v_n^{(L)} \\ 0 \le v_n^{(L)} \perp w_n^{(L)} \ge 0 \\ 0 \le v_n^{(R)} \perp w_n^{(R)} \ge 0 \end{cases}$$

from which, after putting for shortness $f_n = -\bar{Z}_n - F\bar{Y}_n + FD_n e$, we obtain

$$\left\{ \begin{array}{l} \left(\begin{array}{c} w_n^{(L)} \\ w_n^{(R)} \end{array} \right) = \left(\begin{array}{c} f_n \\ e \end{array} \right) + \left(\begin{array}{cc} -2FD_n & I \\ -I & 0 \end{array} \right) \left(\begin{array}{c} v_n^{(L)} \\ v_n^{(R)} \end{array} \right) \\ 0 \le v_n^{(L)} \perp w_n^{(L)} \ge 0 \\ 0 \le v_n^{(R)} \perp w_n^{(R)} \ge 0 \end{array} \right. . \tag{25}$$

To study the solvability of the LCP (25) which is equivalent to the original problem (23) we consider the following result.

Proposition 5.3. *Let $K + \mu d_M(t) \ge 0$ and $r > 0$. For h sufficiently small such that (22) holds the LCP (25) admits a solution.*

Proof. It is elementary to verify that

$$FD_n = \frac{h(K + \mu d_M(t))}{1 - hr} \left(\begin{array}{cc} 0_{q\times q} & 0_{q\times m} \\ 0_{m\times q} & T \end{array} \right),$$

where

$$T = -C \left(h^\alpha \left(I_q - h^\alpha A \right)^{-1} B + B \right) \left(C \left(I - h^\alpha A \right)^{-1} B \right)^{-1},$$

and the application of Lemma 5.1 leads to

$$FD_n = \frac{h(K + \mu d_M(t))}{1 - hr} \left(\begin{array}{cc} 0_{q\times q} & 0_{q\times m} \\ 0_{m\times q} & -I_{m\times m} \end{array} \right).$$

Thus, the matrix $-2FD_n$, together with the coefficients matrix of (25), is a positive semi–definite matrix and the LCP (25) is solvable (e.g., see [11]). □

Once the LCP (25) has been solved and the value of $v_n^{(L)}$ is available, the solution of the original problem (23) can be directly evaluated as

$$\bar{y}_n = \bar{Y}_n + D_n x_n, \quad x_n = 2v_n^{(L)} - e$$

and the corresponding values for the sliding surface and the control are obtained by means of

$$\left\{ \begin{array}{l} \sigma_n = \bar{Z}_n + h^{1-\alpha} C y_n \\ u_n = -(CB)^{-1} CA y_n + r(CB)^{-1} \sigma_n - (K + \mu d_M(t))(CB)^{-1} \operatorname{Sgn}(\sigma_n) \end{array} \right. .$$

6. Numerical simulation

In this Section we present a numerical simulation for the system described in Section 5 with the aim of verifying the free–chattering behavior of the FAM method (11). To facilitate the validation of the results presented in this paper we will use the same coefficients used for the numerical simulation in [39] which are related to the model studied in [32] of an aluminum rod heated from one of its sides.

Thus, let us consider for $q = 3$ and $m = 1$ the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.0601251 & -0.42833 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 6 & 5 & 1 \end{pmatrix}$$

and assume the value $\alpha = 0.5$ for the fractional order. The disturbance $d(t) = 1 + \sin(t)$ is bounded by the constant bound $d_M(t) = 2$.

The numerical simulation has been carried out in Matlab ver. 7.7 and the LCP has been solved by means of the Matlab LCP code available in the file exchange service of Matlab central at `http://www.mathworks.com`.

As we can observe from Figure 5, where the time evolution of the system along the interval $[0, 1]$ is shown in the first plot, the state variables reach the steady state in a continuous way, without the noticeable chattering behavior exhibited by $y_3(t)$ in [39] by using the same control. A relatively large value of $h = 2^{-5}$ has been used in the experiment.

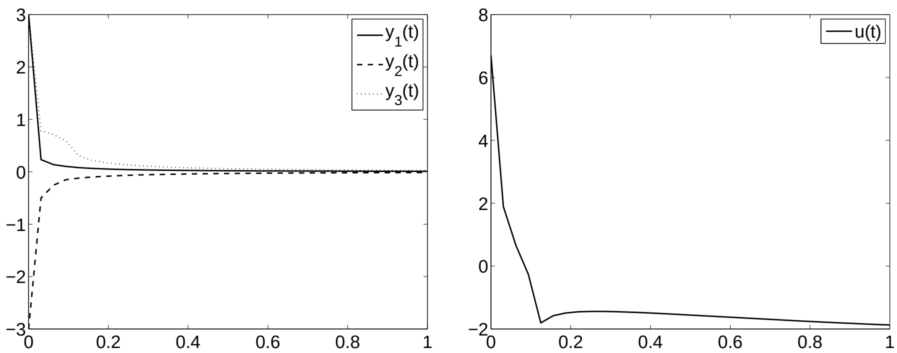

Figure 5: State variables $y_1(t)$, $y_2(t)$ and $y_3(t)$ (left) and control input $u(t)$ (right).

The absence of unwanted oscillations is confirmed by the behavior of the control input $u(t)$ in the second plot in Figure 5.

Furthermore, the chattering–free behavior is better observable from Figure 6 where we draw the history of the sliding variable $\sigma(t)$. As requested, the sliding surface is reached in a finite time and from that instant the dynamics of the system moves along the sliding surface. As we can see from the second plot in Figure 6, where the graph of $|\sigma(t)|$ has been plotted in the logarithmic scale, there are not any hidden oscillations and the system actually evolves on the sliding surface within the numerical accuracy of the method.

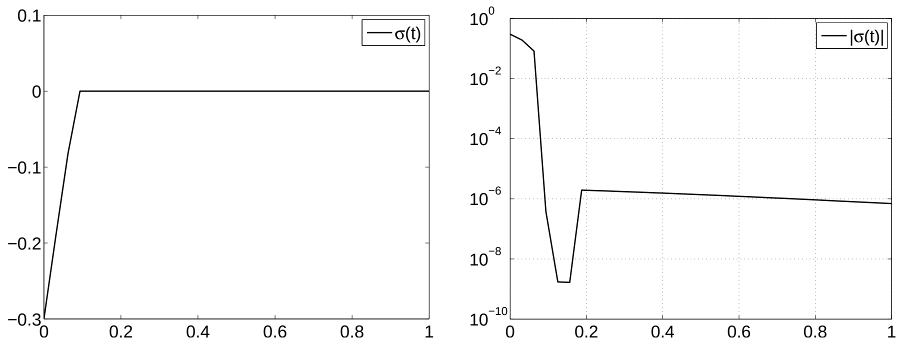

Figure 6: Sliding variable $\sigma(t)$ (left) and its absolute value in logarithmic scale (right).

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