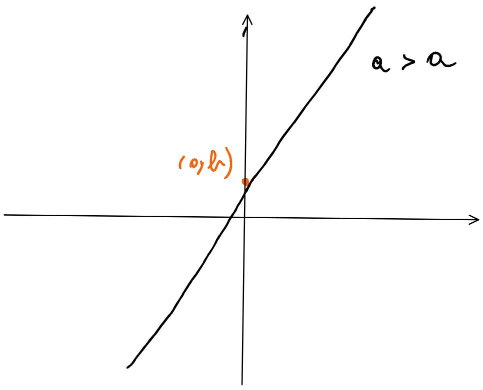
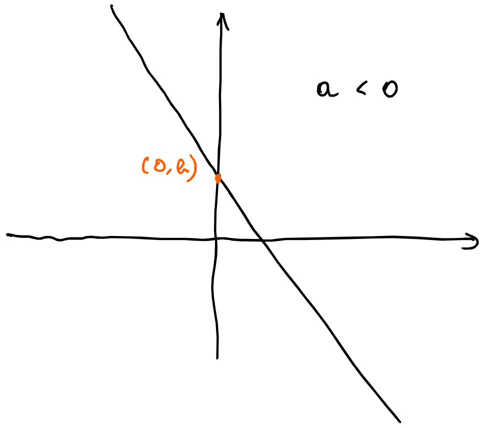
LABORATORIO DI MATEMATICA 4

martedì 10 febbraio 2026 09:39

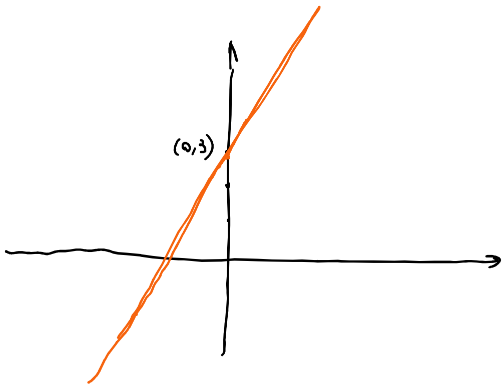
Funzioni elementari:

· $f(x) = ax + b$ (il grafico è una retta)

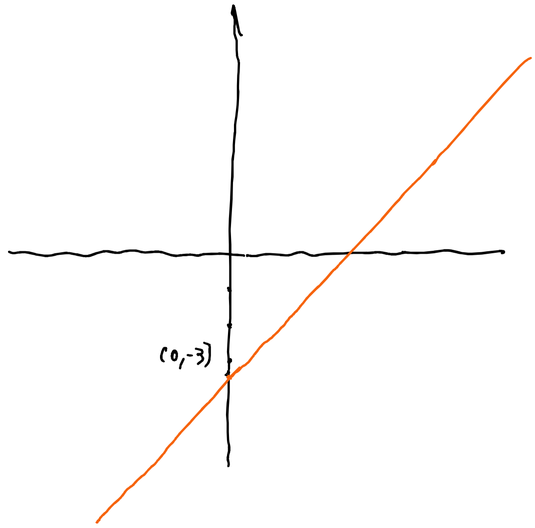




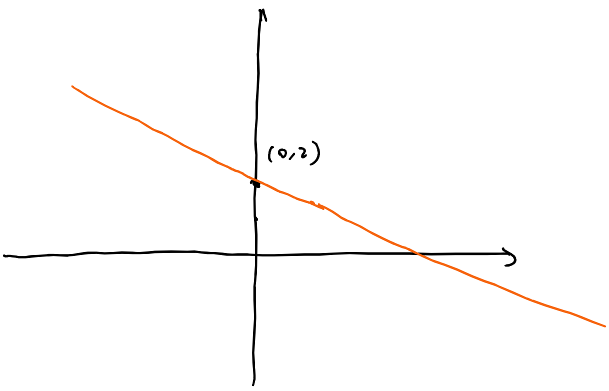
$f(x) = 2x + 3$



$f(x) = x - 3$



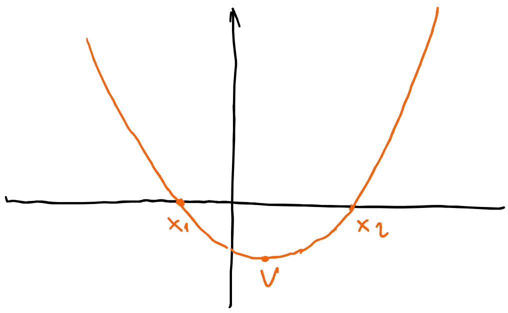
$f(x) = 2 - \frac{1}{3}x$



2) $f(x) = ax^2 + bx + c$

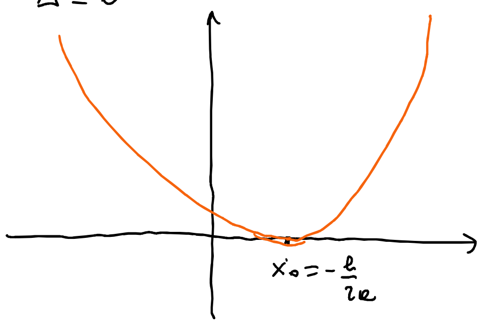
- $a > 0$

$\Delta > 0$: $x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$



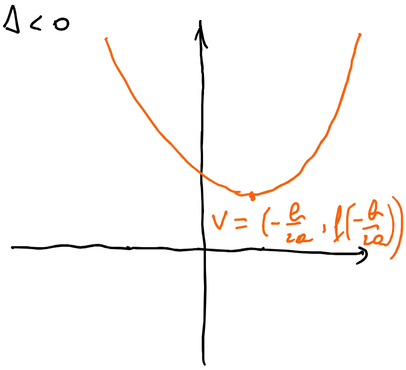
$V = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

$\Delta = 0$



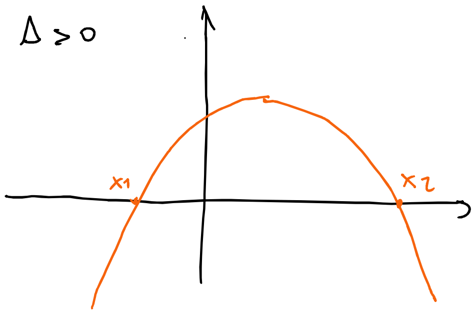
$V = \left(-\frac{b}{2a}, 0\right)$

$\Delta < 0$

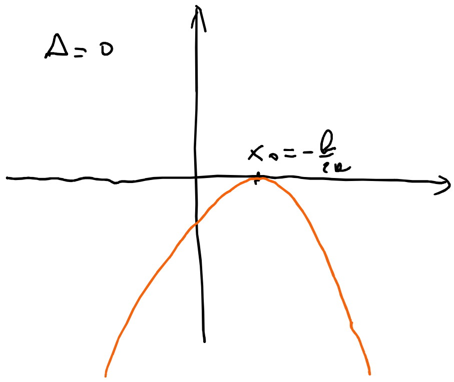


$a < 0$

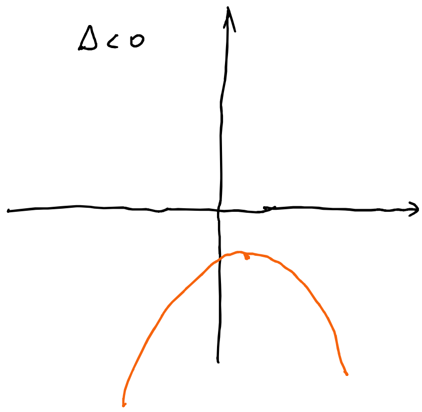
$\Delta > 0$



$\Delta = 0$

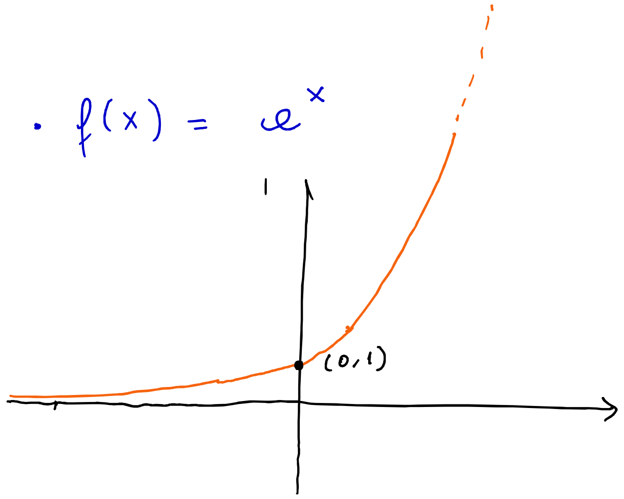


$\Delta < 0$



3) Funzione esponenziale

- $f(x) = e^x$



- $\lim_{x \to +\infty} e^x = +\infty$
- $\lim_{x \to -\infty} e^x = 0$
- $e^x > 0 \quad \forall x \in \mathbb{R}$.
- $e^{f(x)} > 0 \quad \forall x \in \mathrm{Dom}(f)$

- $e^x = 0$ impossibile ($\nexists x \in \mathbb{R}$ t.c. $e^x = 0$)
- $e^x = a < 0$ impossibile ($\nexists x \in \mathbb{R}$ t.c. $e^x = a$)

- $f(x) = e^x$ per $x \to +\infty$ cresce più rapidamente di tutte le potenze.

$$\lim_{x \to +\infty} \frac{e^x}{x} = +\infty, \quad \lim_{x \to +\infty} \frac{e^x}{x^2} = +\infty, \quad \lim_{x \to +\infty} \frac{e^x}{x^{10}} = +\infty.$$

• Attenzione, nei limiti l'esponente è importante.

$$\lim_{x \to +\infty} \frac{e^{x}}{x^{20}} = +\infty$$

$$\lim_{x \to +\infty} \frac{e^{-x}}{x^{20}} = \frac{e^{-\infty}}{+\infty} = \frac{0}{+\infty} = 0 \qquad -x \to -\infty$$

$$\lim_{x \to +\infty} \frac{e^{\frac{1}{x}}}{2-x} = \frac{1}{-\infty} = -\frac{1}{\infty} = 0$$

$$\frac{1}{x} \xrightarrow{x \to +\infty} \frac{1}{+\infty} = 0$$

$$e^{\frac{1}{x}} \xrightarrow{x \to +\infty} e^{0} = 1$$

• Ricordare

$$\boxed{\lim_{x \to +\infty} \frac{e^{\alpha x^{\beta}}}{x^{\gamma}} = +\infty \quad \text{con } \alpha, \beta, \gamma > 0}$$

• $e^{x} = 1 \iff x = 0$

• $e^{x} = a > 0 \iff x = \ln a$.

$\geq \qquad \geq$

$\leq \qquad \leq$

$> \qquad >$

$< \qquad <$

• $$\lim_{x \to +\infty} \frac{e^{x^2}}{x^{10} - 1} = \lim_{x \to +\infty} \frac{e^{x^2}}{x^{10}} = +\infty.$$

• $$\lim_{x \to +\infty} \frac{e^{-2x^2}}{x^4 - x^3 + 2} = \lim_{x \to +\infty} \frac{e^{-2x^2}}{x^4} = \text{“}\frac{e^{-\infty}}{+\infty}\text{”} = \frac{0}{+\infty} = 0$$

• $$\lim_{x \to +\infty} \frac{e^{2x}}{e^{x} - 1} = \lim_{x \to +\infty} \frac{e^{2x}}{e^{x}} = \lim_{x \to +\infty} e^{2x - x} = \lim_{x \to +\infty} e^{x} = +\infty.$$

• $$\lim_{x \to -\infty} \frac{e^{x}}{x} = \frac{0}{-\infty} = 0$$

• $$\lim_{x \to -\infty} \frac{e^{-x}}{x} \overset{y=-x}{=} \lim_{y \to +\infty} \frac{e^{y}}{-y} = \lim_{y \to +\infty} -\frac{e^{y}}{y} = -\infty \qquad \frac{e^{y}}{y} \to +\infty$$

• $$\lim_{x \to 0} \frac{e^{x}}{x^{2}} = \frac{e^{0}}{0^{+}} = \frac{1}{0^{+}} = +\infty.$$

Logaritmo naturale

$f(x) = \ln x$

- $\mathrm{Dom}(f) = \,]0, +\infty[\qquad (x > 0).$

Nota:

$f(x) = \ln g(x)$

Per determinare il dominio di f bisogno risolvere $g(x) > 0$.

- $f(x) = \ln(x-3)$

  $x - 3 > 0 \iff x > 3$

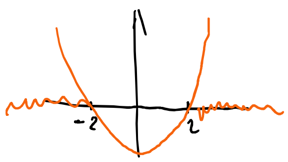
  $\mathrm{Dom}(f) = \,]3, +\infty[.$

- $f(x) = \ln(x^2 - 4)$

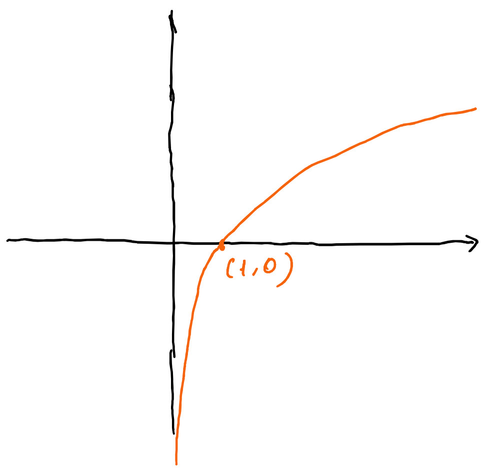
  $x^2 - 4 > 0$

  $x < -2 \quad \vee \quad x > 2$

  ($x^2 - 4$ è un pol. di 2° grado che si annulla in ± 2



- Grafico:



$\ln 1 = 0$

$\lim_{x \to +\infty} \ln x = +\infty$

$\lim_{x \to 0} \ln x = -\infty$

- $\ln x$ cresce più lentamente di x e di tutte le potenze positive di x.

$$\lim_{x \to +\infty} \frac{\ln x}{x} = 0 \quad e \quad \lim_{x \to +\infty} \frac{\ln x}{x^\varepsilon} = 0 \quad \forall \varepsilon > 0.$$

ESEMPI

$$\lim_{x\to+\infty} \frac{\ln x}{2x-3} = \lim_{x\to+\infty} \frac{\ln x}{2x} = 0$$

$$\lim_{x\to+\infty} \frac{\ln x}{x^3-x^2+2} = \lim_{x\to+\infty} \frac{\ln x}{x^3} = 0$$

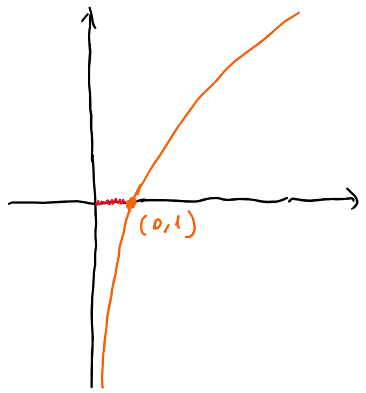
$$\lim_{x\to+\infty} \frac{x}{\ln x} = +\infty$$

(Nota $\lim_{x\to+\infty} \frac{x^\gamma}{e^{\alpha x^\beta}} = 0$ $\forall\, \alpha, \beta, \gamma > 0$)

$$\lim_{x\to+\infty} \frac{\ln x}{3\ln x + 2} = \lim_{x\to+\infty} \frac{\cancel{\ln x}}{3\cancel{\ln x}} = \frac{1}{3}$$

$$\lim_{x\to+\infty} \frac{\ln^2 x}{\ln x \underbrace{-7\ln^2 x}_{\text{Termine principale}} + 4} = \lim_{x\to+\infty} \frac{\ln^2 x}{-7\ln^2 x} = -\frac{1}{7}$$

- $\ln x = 0 \iff x = 1$
- $\ln x > 0 \iff x > 1$
- $\ln x < 0 \iff 0 < x < 1$
- $\ln x \geq 0 \iff x \geq 1$
- $\ln x \leq 0 \iff 0 < x \leq 1$



- $\ln x = c \iff x = e^c$
- $\ln x \geq c \iff x \geq e^c$
- $\ln x \leq c \iff 0 < x \leq e^c$

ESEMPIO

$$f(x) = \frac{e^{2x}}{\ln x + 1}$$

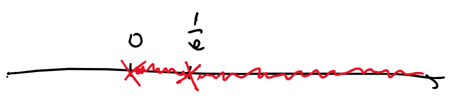
Attenzione
$\ln x + 1 = (\ln x) + 1 = 1 + \ln x$
da non confondere con $\ln(x+1)$

- Dominio :

$x > 0$ e $\ln x + 1 \neq 0$

$\ln x \neq -1$

$x \neq e^{-1}$



$$x \neq \frac{1}{e}$$

$$\mathrm{Dom}(f) =]0,+\infty[\setminus \left\{\frac{1}{e}\right\} =]0,\tfrac{1}{e}[\cup]\tfrac{1}{e},+\infty[$$

- Il grafico di f interseca l' asse x ?

$$\frac{e^{2x}}{\ln x + 1} = 0 \iff e^{2x} = 0 \quad \text{impossibile.}$$

- Il grafico di f interseca l'asse y ?

Bisogna controllare se $0 \in \mathrm{Dom}(f)$ o no.

1) Se $0 \in \mathrm{Dom}(f)$, l'intersezione è $(0, f(0))$
2) Se $0 \notin \mathrm{Dom}(f)$, non ci sono intersezioni del grafico con l'asse y.

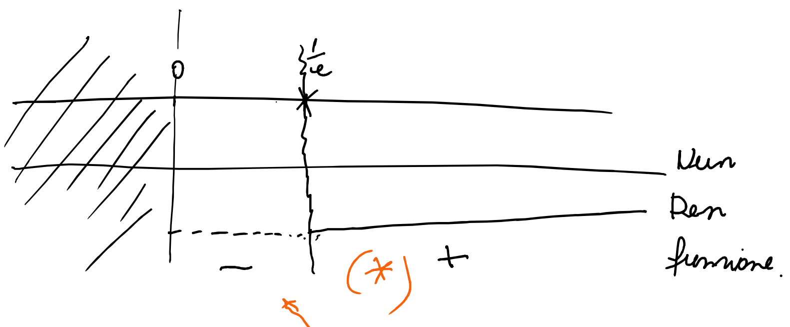
Nel nostro caso
$\mathrm{Dom}(f) =]0,\frac{1}{e}[\cup]\frac{1}{e}, +\infty[$.
$0 \notin \mathrm{Dom}(f) \Rightarrow$ non ci sono intersezioni con l' asse x.

- Segno di f

$$f(x) = \frac{e^{2x}}{\ln x + 1}.$$

Numeratore:
$e^{2x} > 0 \quad \forall x \in \mathbb{R}.$

Denominatore:
$\ln x + 1 > 0$
$\ln x > -1$
$x > e^{-1}$
$x > \frac{1}{e}$

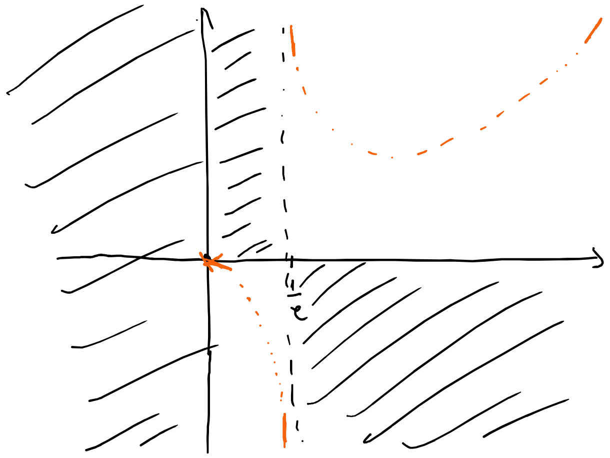


- $\displaystyle \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^{2x}}{1+\ln x} = \frac{1}{-\infty} = -\frac{1}{\infty} = 0$

- $\displaystyle \lim_{x\to \frac{1}{e}^-} f(x) = \lim_{x\to\frac{1}{e}} \frac{e^{2x}}{1+\ln x} = \frac{e^{\frac{2}{e}}}{0^-} = -\infty$

$\displaystyle \lim_{x\to\frac{1}{e}^+} f(x) = +\infty$ * per il segno della funzione.

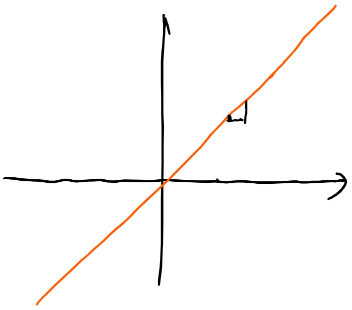
- $\lim_{x \to +\infty} \frac{e^{2x}}{1 + \ln x} = \lim_{x \to +\infty} \frac{e^{2x}}{\ln x} = +\infty$ perché e^{2x} cresce più rapidamente.



Per completare lo studio della funzione serve la derivata.
La derivata di una funzione $f(x)$ è una seconda funzione che ci dice qual è la pendenza del grafico di f in ogni punto.

Derivata.

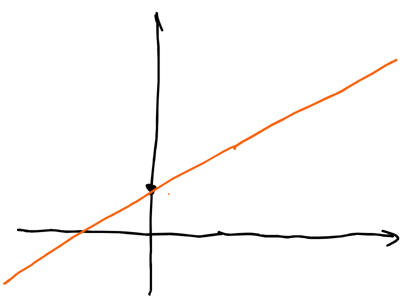
$f(x) = x$



La pendenza del grafico è uguale a 1 in tutti i punti.

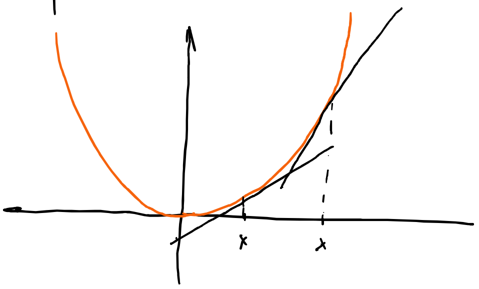
Scriveremo che $f'(x) = 1 \quad \forall x \in \mathbb{R}$.

$f(x) = ax + b$



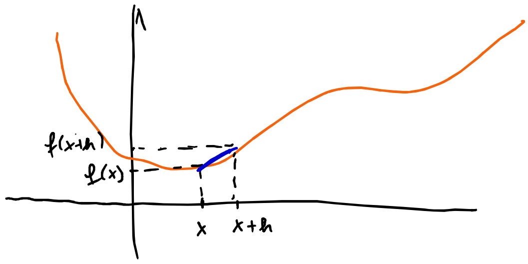
La pendenza è a in tutti i punti cioè $f'(x) = a$.

$f(x) = x^2$



Si può far vedere che $f'(x) = 2x$.

Idea per definire la derivata



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Se $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} 2x + h = 2x$$

Nota: La derivata di $f(x)$ si può indicare con $f'(x)$, $Df(x)$, $\frac{d}{dx} f(x)$.

Derivate delle funzioni elementari:

- $f(x) = 1 \Rightarrow Df(x) = 0 \quad (f'(x) = 0)$
- $f(x) = c$ (costante) $\Rightarrow Df(x) = 0$
- $f(x) = x \Rightarrow Df(x) = 1$
- $f(x) = ax + b \Rightarrow Df(x) = a$
- $f(x) = x^\alpha \Rightarrow Df(x) = \alpha x^{\alpha - 1}$ $\quad \left(\begin{array}{l} f(x) = x^2 \\ Df(x) = 2x^{2-1} = 2x^1 = 2x \end{array} \right)$
- $f(x) = e^x \Rightarrow Df(x) = e^x$
- $f(x) = \ln x \Rightarrow Df(x) = \frac{1}{x}$
- $f(x) = \ln |x| \Rightarrow Df(x) = \frac{1}{x}$

- $f(x) = \sin x \Rightarrow Df(x) = \cos x$
- $f(x) = \cos x \Rightarrow Df(x) = -\sin x$
- $f(x) = \tan x \Rightarrow Df(x) = \frac{1}{\cos^2 x} = 1 + \tan^2 x$
- $f(x) = \arctan x \Rightarrow Df(x) = \frac{1}{x^2+1}$

Regole di derivazione elementari

1) $D(f \pm g) = Df \pm Dg$ (DERIVATA DI SOMME / DIFFERENZE)

2) $D(cf) = c\,Df \quad \forall c \in \mathbb{R}$ (DERIVATA DI UN MULTIPLO)

3) $D(f \cdot g) = Df \cdot g + f \cdot Dg.$ (DERIVATA DEL PRODOTTO)

4) $D\left(\frac{f}{g}\right) = \frac{Df \cdot g - f \cdot Dg}{g^2}$ (DERIVATA DI UN RAPPORTO)

5) $D\left(\frac{1}{g}\right) = -\frac{Dg}{g^2}$ (DERIVATA DEL RECIPROCO)

6) $D(g \circ f) = (Dg \circ f) \cdot Df.$

cioè $D(g(f(x))) = Dg(f(x)) \cdot Df(x).$

ESEMPI

1) $f(x) = x^7 - x^6$

$Df(x) = Dx^7 - Dx^6 = 7x^6 - 6x^5$

2) $f(x) = 2x^3 + \pi x^5$

$$\begin{aligned} Df(x) &= 2\,Dx^3 + \pi\,Dx^5 \\ &= 2 \cdot 3x^2 + \pi \cdot 5x^4 \\ &= 6x^2 + 5\pi x^4. \end{aligned}$$

3) $D(2x^2 - 3x + 2) = 4x - 3$

4) $D(x^2 e^x)$

$$= D x^2 \cdot e^x + x^2 D e^x$$
$$= 2x\, e^x + x^2 e^x .$$

$$\left(\begin{aligned} &= e^x (2x + x^2) \\ &= x e^x (2+x) \end{aligned} \right)$$

5) $$D\left(\frac{x}{\sin x}\right) = \frac{Dx \cdot \sin x - x\, D \sin x}{\sin^2 x}$$

$$= \frac{1 \cdot \sin x - x \cdot \cos x}{\sin^2 x}$$

$$= \frac{\sin x - x \cos x}{\sin^2 x} .$$

$$D\left(\frac{e^x}{1+e^x}\right) = \frac{D e^x \cdot (1+e^x) - e^x D(1+e^x)}{(1+e^x)^2}$$

$$= \frac{e^x (1+e^x) - e^x \cdot e^x}{(1+e^x)^2}$$

$$= \frac{e^x (1 + \cancel{e^x} - \cancel{e^x})}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$$

$$D \frac{1}{x} = D x^{-1} = -1 \cdot x^{-2} = -\frac{1}{x^2} \qquad \left(\text{o } D \frac{1}{x} = \frac{-Dx}{x^2} = -\frac{1}{x^2} \right)$$

$$D \sqrt{x} = D x^{\frac{1}{2}} = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

Ricordare

$$\boxed{D \frac{1}{x} = -\frac{1}{x^2} \qquad \text{e} \qquad D\sqrt{x} = \frac{1}{2\sqrt{x}}}$$

$$D \frac{1}{\sqrt[3]{x^4}} = D x^{-\frac{4}{3}} = -\frac{4}{3} x^{-\frac{4}{3}-1} = -\frac{4}{3} x^{-\frac{7}{3}} = -\frac{4}{3} \frac{1}{\sqrt[3]{x^7}}$$

Derivate di composizioni

$$D(g(f(x))) = D g(f(x)) \cdot D f(x)$$

ESEMPI

$$D\, e^{x^2} = D g(f(x)) \cdot Df(x) = e^{x^2} \cdot 2x$$

$g(x) = e^x$, $f(x) = x^2$

$Dg(x) = e^x$

$Dg(f(x)) = e^{f(x)} = e^{x^2}$.

$Df(x) = 2x$.

Casi particolari di composizioni

$$D\, e^{f(x)} = e^{f(x)} \cdot Df(x).$$

$$D(\ln f(x)) = \frac{1}{f(x)} \cdot Df(x)$$

$$D(\sin f(x)) = \cos(f(x)) \cdot Df(x)$$

$$D(\cos f(x)) = -\sin(f(x)) \cdot Df(x)$$

$$D\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}} \cdot Df(x)$$

ESEMPI

$$D\, e^{2x} = e^{2x} \cdot D(2x) = e^{2x} \cdot 2 = 2e^{2x}$$

$$D\, e^{\cos x} = e^{\cos x} \cdot D(\cos x) = e^{\cos x} \cdot (-\sin x) = -e^{\cos x} \sin x$$

$$D\, \frac{e^{2x}}{x^2-1} = \frac{D(e^{2x}) \cdot (x^2-1) - e^{2x}\, D(x^2-1)}{(x^2-1)^2} = \frac{e^{2x} \cdot 2 \cdot (x^2-1) - e^{2x} \cdot 2x}{(x^2-1)^2} = \frac{2e^{2x}(x^2-1-x)}{(x^2-1)^2}$$

$$D\, \frac{e^{2x}}{e^x-1} = \frac{D(e^{2x}) \cdot (e^x-1) - e^{2x} \cdot D(e^x-1)}{(e^x-1)^2} = \frac{e^{2x} \cdot 2 \cdot (e^x-1) - e^{2x} \cdot e^x}{(e^x-1)^2}$$

$$= \frac{e^{2x}\left(2(e^x-1) - e^x\right)}{(e^x-1)^2}$$

$$= \frac{e^{2x}\left(\cancel{2}e^x - 2 - \cancel{e^x}\right)}{(e^x-1)^2}$$

$$= \frac{e^{2x}\left(e^x - 2\right)}{(e^x-1)^2}$$

$$D\,\frac{e^{\frac{1}{x}}}{2-x} = \frac{D\,e^{\frac{1}{x}}\cdot(2-x) - e^{\frac{1}{x}}\cdot D(2-x)}{(2-x)^2}$$

$$= \frac{e^{\frac{1}{x}}\,D\left(\frac{1}{x}\right)\cdot(2-x) - e^{\frac{1}{x}}(-1)}{(2-x)^2}$$

$$= \frac{e^{\frac{1}{x}}\left(-\frac{1}{x^2}(2-x)\right) + e^{\frac{1}{x}}}{(2-x)^2}$$

$$= \frac{e^{\frac{1}{x}}\left(-\frac{2-x}{x^2}+1\right)}{(2-x)^2} = \frac{e^{\frac{1}{x}}\left(\frac{-2+x+x^2}{x^2}\right)}{(2-x)^2}$$

$$= \frac{e^{\frac{1}{x}}\left(x^2+x-2\right)}{x^2(2-x)^2}$$

- $D\ln(x^3+2x+1)$

$$= \frac{1}{x^3+2x+1}\cdot D(x^3+2x)$$

$$= \frac{3x^2+2}{x^3+2x+1}$$

- $D\sqrt{x^2-4} = \frac{1}{2\sqrt{x^2-4}}\,D(x^2-4)$

$$= \frac{1}{\cancel{2}\sqrt{x^2-4}}\cdot\cancel{2}x = \frac{x}{\sqrt{x^2-4}}$$

<u>Nota</u>: Il segno della derivata consente di studiare la monotonia della funzione.

$f(x) = x^3 - 2x$

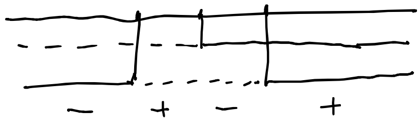
- $\mathrm{Dom}(f) = \mathbb{R}$.
- $0 \in \mathrm{Dom}(f) \Rightarrow f(0) = 0$. $(0,0)$ è intersezione con asse y.
- Asse x:

$$x^3 - 2x = 0 \iff x(x^2-2) = 0$$
$$\iff x = 0 \ \vee \ x^2 - 2 = 0$$
$$\iff x = 0 \ \vee \ x = \pm\sqrt{2}$$

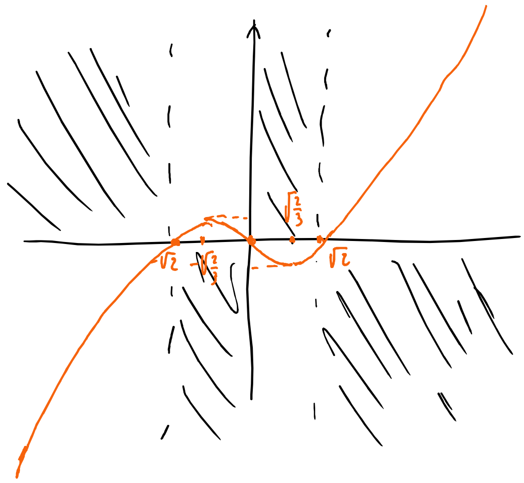
- Segno:

$x^3 - 2x > 0$

$x(x^2-2)$



- $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} x^3 - 2x = +\infty$
- $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} x^3 - 2x = -\infty$.



$Df(x) = D(x^3 - 2x) = 3x^2 - 2$

Segno di $Df(x)$: