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A NOTE ON COHEN-MACAULAY GRAPHS

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ABSTRACT. We show that the edge ideal of a Cohen-Macaulay graph on $2n$ non-isolated vertices, whose height is n, is always a set-theoretic complete intersection. This result, in particular, applies to all Cohen-Macaulay bipartite graphs.

1. INTRODUCTION

Let K be a field, and let V be a set of indeterminates over K. Let I be a homogeneous ideal of the polynomial ring $R = K[V]$. As a consequence of the graded Auslander-Buchsbaum formula, the equality $\mathrm{pd}(R/I) = \mathrm{ht}\, I$, where pd denotes the projective dimension and ht the height, holds if and only if the quotient ring R/I is Cohen-Macaulay. On the other hand, if I is generated by squarefree monomials, we have the following well-known inequalities:

$$\mathrm{ht}\, I \leq \mathrm{pd}(R/I) \leq \mathrm{ara}\, I, \tag{0}$$

where ara denotes the arithmetical rank, i.e., the minimum number of elements of R that generate an ideal whose radical is I. Recall that an ideal is called a set-theoretic complete intersection whenever its arithmetical rank equals its height. If I is generated by squarefree monomials, in view of the above remarks, this condition implies that the quotient ring R/I is Cohen-Macaulay. Furthermore, it is well-known that, if R/I is Cohen-Macaulay, then I is unmixed.

Now suppose that the generators of I are squarefree quadratic monomials. In this case I can be associated with a graph G on the vertex set $V(G) = V$, whose edge set $E(G)$ is formed by all subsets $\{u, v\}$ of V such that $uv \in I$. (We will write uv for $\{u, v\}$.) In this setting, I is called the *edge ideal* of G and is denoted by $I(G)$. Its minimal monomial generators are called the *edge monomials* of G. The idea of introducing this notion is due to Villarreal [7], and dates back to 1990. Since then, edge ideals have been the object of thorough investigations concerning the connections between their graph-theoretic and their ring-theoretic properties. In [1], Crupi, Rinaldo and Terai considered the graphs G on $2n$ non-isolated vertices for

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which $I(G)$ has height n. They gave a combinatorial criterion for the unmixedness of $I(G)$, which, in this case, implies (hence, is equivalent to) the Cohen-Macaulay property of the quotient ring $R/I(G)$. It is reported here as Theorem 3.2 and consists of three conditions, which generalize those given by Herzog and Hibi [3] for bipartite graphs. In this paper we shed a new light on this result by proving that the edge ideal of a graph fulfilling these three conditions is always a set-theoretic complete intersection. The main tool used here is a theorem due to Kimura [4], which gives an upper bound of the arithmetical rank of a monomial ideal in terms of certain divisibility relations on the set of generators.
All the results quoted or proven here are independent of the field K, a fact that emphasizes their strongly combinatorial nature.

2. Preliminaries

Let I be an ideal of R generated by monomials, and let $S : \alpha_1, \ldots, \alpha_r$ be an ordered sequence of its minimal monomial generators. The following definition can be given for an arbirtrary monomial ideal of R. In brackets we will add an equivalent formulation for the case where I is generated by squarefree monomials of degree two.

Definition 2.1. For all subsequences $\alpha_{i_1}, \ldots, \alpha_{i_t}$ of S, we set

$$L(\alpha_{i_1}, \ldots, \alpha_{i_t}) = \{\alpha_{i_1}, \ldots, \alpha_{i_t}\},$$

and we call it an *admissible symbol* of *dimension* s if α_q does not divide $\mathrm{lcm}(\alpha_{i_h}, \alpha_{i_{h+1}}, \ldots, \alpha_{i_t})$ for any $h < t$ such that $q < i_h$ (i.e., equivalently, if α_q does not divide any product $\alpha_{i_h}\alpha_{i_k}$ for any h, k such that $h < k \leq t$ and $q < i_h$).

Set $L_0 = R$ and for all $t = 1, \ldots, r$, let L_t be the free R-module generated by all admissible symbols of dimension t. Define the map $d_t : L_t \to L_{t-1}$ by setting

$$d_t(L(\alpha_{i_1}, \ldots, \alpha_{i_t})) =$$
$$\sum_{j=1}^{t} (-1)^{j+1} \frac{\mathrm{lcm}(\alpha_{i_1}, \ldots, \alpha_{i_t})}{\mathrm{lcm}(\alpha_{i_1}, \ldots, \widehat{\alpha_{i_j}}, \ldots, \alpha_{i_t})} L(\alpha_{i_1}, \ldots, \widehat{\alpha_{i_j}}, \ldots, \alpha_{i_t}).$$

Then one has the following

Theorem 2.2. ([6], p. 193) *The complex*

$$0 \to L_r \overset{d_r}{\to} L_{r-1} \overset{d_{r-1}}{\to} \cdots \overset{d_1}{\to} L_0 \to 0 \qquad (\star)$$

is a free resolution of R/I.

The resolution $(\star)$ is called a *Lyubeznik resolution* of I. Note that the Lyubeznik resolution of I in general strictly depends on the order of the

sequence $\alpha_1, \dots, \alpha_r$: different permutations of the α_i can give rise to non-isomorphic resolutions.
Also note that any subset of an admissible symbol is again admissible.

We now recall a crucial result due to Kimura:

Theorem 2.3. ([4], Theorem 1) *Let I be a monomial ideal of R. If I has a Lyubeznik resolution of length ℓ, then* $\operatorname{ara} I \leq \ell$.

We will say that an admissible symbol is *maximal* (with respect to S) if it is of maximum dimension among the admissible symbols.

3. On a special class of Cohen-Macaulay graphs

We recall that a graph G is called Cohen-Macaulay if the quotient ring $R/I(G)$ is Cohen-Macaulay for all fields K. It is called unmixed if so is the ideal $I(G)$ (over every field K).
Let G be an unmixed graph on $2n$ vertices $x_1, \dots, x_n, y_1, \dots, y_n$, all non-isolated, and such that $\operatorname{ht} I(G) = n$. Since G has a perfect matching (see [1], Lemma 2.1), we may assume that

($*$) $X = \{x_1, \dots, x_n\}$ is a minimal vertex cover of G and $Y = \{y_1, \dots, y_n\}$ is a maximal independent set of G such that $\{x_1y_1, \dots, x_ny_n\} \subset E(G)$. (Since Y is an independent set, it follows that there are no edges of the form y_iy_j.)

In addition, if G is Cohen-Macaulay (see [1], Section 3), we may also assume that

($**$) if $x_iy_j \in E(G)$, then $i \leq j$.

Definition 3.1. Let G be a graph without isolated vertices and such that all maximal independent sets have the same cardinality. If furthermore this cardinality is n, where $2n$ is the number of vertices of G, then G is called *very well covered*.

The very well covered graphs were studied in [2]. The next result by Crupi, Rinaldo and Terai extends Theorem 3.4 of [3].

Theorem 3.2. ([1], Theorem 3.6) *Let G be a graph without isolated vertices. Let $V(G) = \{x_1, \dots, x_n, y_1, \dots, y_n\}$ and suppose that* $\operatorname{ht} I(G) = n$. *Also assume that G fulfils conditions ($*$) and ($**$). Then the following conditions are equivalent:*

(1) *G is Cohen-Macaulay,*
(2) *G is unmixed,*
(3) *the following conditions hold:*
   (i) *if $z_ix_j, y_jx_k \in E(G)$, then $z_ix_k \in E(G)$ for distinct i, j, k and for $z_i \in \{x_i, y_i\}$,*
   (ii) *if $x_iy_j \in E(G)$, then $x_ix_j \notin E(G)$.*

Remark 3.3. Note that conditions ($*$), ($**$) and (3) are preserved when we remove a set of vertices of the form $\{x_i, y_i \mid i \in S\}$ with $S \subset \{1, \dots, n\}$.

Theorem 3.4. *Let G be a very well covered graph on $2n$ vertices $x_1, \dots x_n$, $y_1, \dots, y_n$ fulfilling conditions $(*)$ and $(**)$. Then* $\operatorname{ara} I(G) = \operatorname{ht} I(G) = n$, *i.e., $I(G)$ is a set-theoretic complete intersection.*

Proof. Suppose that G fulfils condition (3). We show that $I(G)$ has a Lyubeznik resolution of length at most n. In view of (0) and Theorem 2.3, this will prove our claim.
Consider the lexicographic order with respect to the following arrangement of the variables:

$$x_1 > x_2 > \cdots > x_n > y_n > y_{n-1} > \cdots > y_1.$$

Arrange the edge monomials of G according to the induced ordering, i.e. the ordering induced by

$$\begin{array}{cccccccccc}
x_1x_2 & x_1x_3 & \cdots & x_1x_n & x_1y_n & x_1y_{n-1} & \cdots & x_1y_2 & x_1y_1 \\
 & x_2x_3 & \cdots & x_2x_n & x_2y_n & x_2y_{n-1} & \cdots & x_2y_2 & \\
 & & \ddots & \vdots & \vdots & \vdots & \iddots & & \\
 & & & x_{n-1}x_n & x_{n-1}y_n & x_{n-1}y_{n-1} & & & \\
 & & & & x_ny_n & & & &
\end{array}$$

We prove that every admissible symbol with respect to this ordering has dimension at most n. We proceed by induction on n. For $n = 1$ the claim is trivial, since, in this case, x_1y_1 is the only edge monomial of G.
Now suppose that $n > 1$ and that the claim is true for all smaller n. Let u be an admissible symbol of maximum dimension and suppose that there are exactly r monomials in u containing the variable x_1, i.e., appearing in the first row of the above table, and precisely $x_1z_{i_1}, x_1z_{i_2}, \dots, x_1z_{i_r}$, where $i_1 < \cdots < i_r$ and $z_{i_h} \in \{x_{i_h}, y_{i_h}\}$ for every h. If $r \le 1$, then the remaining monomials in u form an admissible symbol for a graph of the same type of G on the vertex set $\{x_2, \dots, x_n, y_2, \dots, y_n\}$ (see Remark 3.3). By induction, this symbol has dimension at most $n - 1$, hence u has dimension at most n. Thus suppose that $r \ge 2$. Then every monomial of u lying outside the first row cannot contain any of the variables $z_{i_1}, \dots, z_{i_{r-1}}$, because, for all $h = 1, \dots, r-1$, the monomial $x_1z_{i_h}$ precedes $x_1z_{i_r}$ and divides $x_1z_{i_r}z_{i_h}$.
Let $1 \le h \le r-1$. We want to prove that u does not contain any monomial divisible by $\{x_{i_h}, y_{i_h}\} \setminus \{z_{i_h}\}$.
First suppose that $z_{i_h} = x_{i_h}$. Then $i_h \ne 1$. Moreover, $x_{i_h}y_{i_h} \notin u$, because, as we have just remarked, no monomial of u outside the first row is divisible by x_{i_h}. We prove that u does not contain any monomial divisible by y_{i_h}. Suppose, by contradiction, that $x_ky_{i_h} \in u$ for some k. Then $k \ne i_h$ and, on the other hand, $k \le i_h$ by virtue of condition $(**)$. Moreover, since $x_1x_{i_h} \in E(G)$, from condition (3) (ii) it follows that $k \ne 1$. Finally, since $x_1x_{i_h}, x_ky_{i_h} \in E(G)$, and 1, i_h, k are pairwise distinct, condition (3) (i) implies that $x_1x_k \in E(G)$. Thus u is not admissible, because the monomial x_1x_k precedes both the monomials $x_1x_{i_h}$ and $x_ky_{i_h}$ (since $1 < k < i_h$) and divides their product. This provides a contradiction.

Now suppose that $z_{i_h} = y_{i_h}$. Since $h \leq r-1$, $x_1y_{i_h}$ cannot be the last monomial of the first row of the above table. Hence $i_h \neq 1$. We prove that u does not contain any monomial divisible by x_{i_h}. Suppose, by contradiction, that $x_{i_h}z_k \in u$ for some k. Once again, since $x_{i_h}y_{i_h} \notin u$, we have $k \neq i_h$. Now, $z_k = y_1$ would imply $i_h = 1$ by condition $(**)$, against our assumption. On the other hand, we also have $z_k \neq x_1$, because $x_1y_{i_h} \in E(G)$, which, in view of condition (3) (ii), implies that $x_{i_h}x_1 \notin E(G)$. This proves that $k \neq 1$. Moreover, since $x_1y_{i_h}, x_{i_h}z_k \in E(G)$, from condition (3) (i) it follows that $x_1z_k \in E(G)$. We observe that the monomial x_1z_k precedes $x_1y_{i_h}$ in the above order: if $z_k = x_k$, this is clear; if $z_k = y_k$, then $x_{i_h}y_k \in E(G)$, so that $i_h < k$ by virtue of condition $(**)$. Hence x_1y_k precedes $x_1y_{i_h}$. Further, since $i_h > 1$, the monomial x_1z_k also precedes $x_{i_h}z_k$. But x_1z_k divides $x_1y_{i_h}x_{i_h}z_k$, which implies that u is not admissible and provides a contradiction.
We have thus proven that the monomials in u following $x_1z_{i_1}, x_1z_{i_2}, \ldots, x_1z_{i_r}$, i.e., those lying in the rows below the first one, form an admissible symbol v for a graph of the same type of G on the vertex set

$$\{x_2, \ldots, x_n, y_2, \ldots, y_n\} \setminus \{x_{i_1}, \ldots, x_{i_{r-1}}, y_{i_1}, \ldots, y_{i_{r-1}}\}.$$

By induction, v has length at most $n-1-(r-1) = n-r$. Therefore u has length at most $r+n-r = n$, so that the claim follows. □

Corollary 3.5. *Let G be a very well covered graph. Then the following conditions are equivalent:*

(1) *$R/I(G)$ is Cohen-Macaulay,*
(2) *$I(G)$ is a set-theoretic complete intersection.*

In particular, this equivalence is true if G is a bipartite graph.

Remark 3.6. The arithmetical rank of the edge ideals of some Cohen-Macaulay bipartite graphs had already been computed by Kummini [5].

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