Corso di laurea in Chimica
Esame di
ISTITUZIONI di MATEMATICHE I

17 febbraio 2017

1. Studiare la seguente equazione nel campo complesso
$$\bar{z} - \frac{1}{1+z} = i^{49} + \left(\frac{1+i}{1-i}\right)^2 + \left(\frac{\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}}{\cos\frac{\pi}{14} + i\sin\frac{\pi}{14}}\right)^{28}.$$

2. Studiare il dominio della funzione
$$y = \sqrt{\frac{3^{2x} - 3^{x+1} + 3^{1-|x|} - 1}{\log^2_{\frac{1}{2}} 2|x| + \log_2|2x|}}.$$

3. Studiare la funzione
$$f(x) = \exp(|x+1|\,(2x - x^2).$$
e disegnarne il grafico approssimativo.

4. Determinare gli asintoti della funzione
$$g(x) = \sqrt{\frac{x^4 - 2x^2 - 3}{x^2 - 4}}\,\sin\left(\frac{1}{x}\right).$$

5. Data la funzione
$$h(x) = \cos x \log\frac{1 + \sin x + \cos^2 x}{\sin x},$$
i) calcolare l'insieme delle primitive,

ii) dire, giustificando le risposte, se è integrabile in senso improprio in $]0, \frac{\pi}{2}]$,in $[\frac{\pi}{2}, \pi[$ e in $[3, +\infty)$.

Svolgimento

1. Poiché $\quad i^{49} = i^{4\cdot 12+1} = (i^4)^{12}\cdot i = i$

$$\left(\frac{1+i}{1-i}\right)^2 = \left(\frac{(1+i)^2}{1-i^2}\right)^2 = \left(\frac{\cancel{1}+\cancel{i^2}+2i}{2}\right)^2 = i^2 = -1$$

$$\left(\frac{\cos\frac{\pi}{7}+i\,\mathrm{sen}\frac{\pi}{7}}{\cos\frac{\pi}{14}+i\,\mathrm{sen}\frac{\pi}{14}}\right)^{28} = \cos\frac{\pi}{\cancel{14}}\cdot \cancel{28}^{2} + i\,\mathrm{sen}\frac{\pi}{\cancel{14}}\cdot\cancel{28}^{2} = \cos 2\pi + i\,\mathrm{sen}\,2\pi = 1$$

l'equazione diventa $\quad \bar z + \frac{1}{1+z} = i - \cancel{1} + \cancel{1} \iff \bar z(1+z) - 1 = i + iz$

Posto $z = x+iy$ e poiché $\bar z = x-iy$, $z\cdot\bar z = x^2+y^2$, si ha

$$x - iy + x^2 + y^2 - 1 = i + ix - y \iff$$

$$\begin{cases} x^2+y^2+x-1 = -y \\ -y = x+1 \end{cases} \iff \begin{cases} y = -x-1 \\ x^2+x^2+\cancel{1}+2x+\cancel{x}-\cancel{1} = \cancel{x}+1 \end{cases}$$

$$2x^2+2x-1=0 \qquad x = \frac{-1\pm\sqrt3}{2}$$

$$\Rightarrow \begin{cases} x = \dfrac{-1+\sqrt3}{2} \\ y = \dfrac{+1-\sqrt3}{2} - 1 = \dfrac{-1-\sqrt3}{2} \end{cases} \vee \begin{cases} x = \dfrac{-1-\sqrt3}{2} \\ y = \dfrac{+1+\sqrt3}{2} - 1 = -\dfrac{1}{2} + \dfrac{\sqrt3}{2} \end{cases}$$

L'equazione ha due soluzioni

$$z_1 = \frac{-1+\sqrt3}{2} + i\,\frac{-1-\sqrt3}{2} \qquad z_2 = \frac{-1-\sqrt3}{2} + i\,\frac{-1+\sqrt3}{2}.$$

2. $$\begin{cases} \dfrac{3^{2x} - 3^{x+1} + 3^{1-|x|} - 1}{\log^2_{\frac12} 2|x| + \log_2|2x|} \geq 0 \\ |x| \neq 0 \iff x \neq 0 \end{cases}$$

$N \geq 0 \qquad 3^{2x} - 3^{x+1} + 3^{1-|x|} - 1 \geq 0 \iff$

$$\begin{cases} x \geq 0 \\ 3^{2x} - 3^{x+1} + 3^{1-x} - 1 \geq 0 \end{cases} \vee \begin{cases} x < 0 \\ 3^{2x} - \cancel{3^{x+1}} + \cancel{3^{1+x}} - 1 \geq 0 \end{cases}$$

(2)

$$\begin{cases} x \geq 0 \\ 3^{2x} - 3 \cdot 3^x + \dfrac{3}{3^x} - 1 \geq 0 \end{cases} \quad \vee \quad \begin{cases} x < 0 \\ 3^{2x} - 1 \geq 0 \end{cases}$$

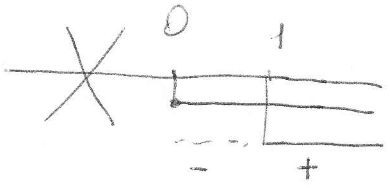
$3^{3x} - 3 \cdot 3^{2x} - 3^x + 3 \geq 0$ $\qquad$ $3^{2x} \geq 1 \qquad 2x \geq \log_3 1 = 0$

$3^{2x}(3^x - 3) - (3^x - 3) \geq 0$ $\qquad$ $\begin{cases} x < 0 \\ x \geq 0 \end{cases} \quad \emptyset$

$(3^{2x} - 1)(3^x - 3) \geq 0$

$3^{2x} \geq 1 \qquad 2x \geq 0$

$3^x \geq 3 \qquad x \geq 1$



Dunque

$N \geq 0 \iff x \geq 1$

$D > 0 \qquad \log^2_{\frac{1}{2}} 2|x| + \log_2 |2x| > 0$

Poiché $\log^2_{\frac{1}{2}} 2|x| = \left(\log_{\frac{1}{2}} 2|x|\right)^2 = \left(\dfrac{\log_2 2|x|}{\log_2 \frac{1}{2}}\right)^2 = \left(-\log_2 2|x|\right)^2$

la disequazione diventa

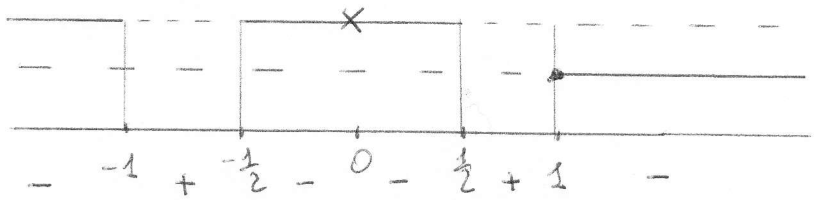
$\log^2_2 |2x| + \log_2 |2x| > 0 \iff \log_2 |2x| < -1 \ \vee \ \log_2 |2x| > 0$

$\iff 0 < |2x| < 2^{-1} \ \vee \ |2x| > 1 \iff 0 < |x| < \frac{1}{2} \ \vee \ |x| > 1$

$\iff x < -1 \ \vee \ -\frac{1}{2} < x < \frac{1}{2} \ \vee \ x > 1$

$x \neq 0$

Conclusione $D > 0$, $N \geq 0$



$ID =]-1, -\frac{1}{2}[\ \cup \ ]\frac{1}{2}, 1]$

3. $ID = \mathbb{R}$ $\qquad f(-x) \neq \begin{matrix} +f(x) \\ -f(x) \end{matrix}$ $\qquad$ f non è né pari né dispari

$\begin{cases} x = 0 \\ y = e^0 = 1 \end{cases} \qquad \begin{cases} y = 0 \\ e^{|x+1|(2x - x^2)} = 0 \end{cases} \quad \emptyset \qquad (0, 1)$ intersezione con asse y

$y > 0 \iff e^{|x+1|(2x - x^2)} > 0 \qquad \forall x \in \mathbb{R}$

$\nexists$ as. verticali perché f è continua in $\mathbb{R}$

$\lim\limits_{x \to \pm\infty} e^{|x+1|(2x - x^2)} = \lim\limits_{y \to -\infty} e^y = 0$ $\quad$ poiché $\lim\limits_{x \to \pm\infty} |x+1|(2x - x^2) = -\infty$ per ordini

$\Rightarrow y = 0$ As. orizz. a $\pm\infty$

$$y' = e^{|x+1|(2x-x^2)}\left(\frac{x+1}{|x+1|}(2x-x^2) + |x+1|(2-2x)\right)$$

$$= e^{|x+1|(2x-x^2)}\,\frac{(x+1)(2x-x^2)+(x+1)^2(2-2x)}{|x+1|}$$

$$= e^{|x+1|(2x-x^2)}\,\frac{x+1}{|x+1|}\left(2x-x^2+2(x+1)(1-x)\right)$$

$$= e^{|x+1|(2x-x^2)}\,\frac{x+1}{|x+1|}(-3x^2+2x+2) \qquad ID(y') = \mathbb{R} - \{-1\}$$

Poiché

$$y' = \begin{cases} e^{(x+1)(2x-x^2)}(-3x^2+2x+2) & x > -1 \\ -e^{-(x+1)(2x-x^2)}(-3x^2+2x+2) & x < -1 \end{cases}$$

si ha

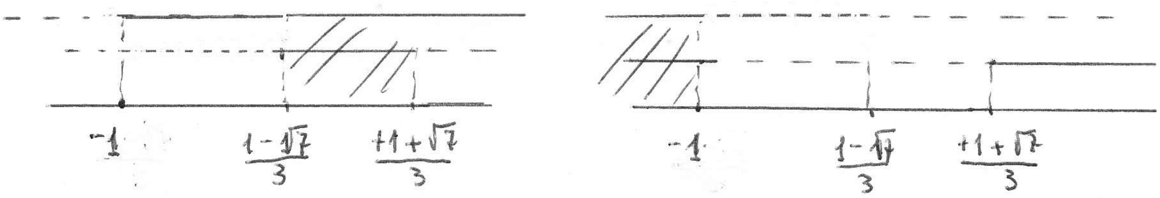
$$\lim_{x\to -1^+} y' = -3 \ \wedge\ \lim_{x\to -1^-} y' = +3 \Rightarrow \begin{cases} x = -1 \\ y = 1 \end{cases} \text{ pto angoloso}$$

essendo $f'_+(-1) = -3 \neq 3 = f'_-(-1)$.

$$y' > 0 \iff \begin{cases} x > -1 \\ -3x^2+2x+2 > 0 \end{cases} \vee \begin{cases} x < -1 \\ -3x^2+2x+2 < 0 \end{cases} \iff$$

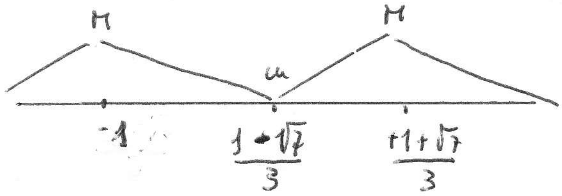
$$x_{1,2} = \frac{+1 \pm \sqrt{7}}{3}$$

$$\begin{cases} x > -1 \\ \frac{+1-\sqrt{7}}{3} < x < \frac{+1+\sqrt{7}}{3} \end{cases} \vee \begin{cases} x < -1 \\ x < \frac{+1-\sqrt{7}}{3} \vee x > \frac{+1+\sqrt{7}}{3} \end{cases}$$



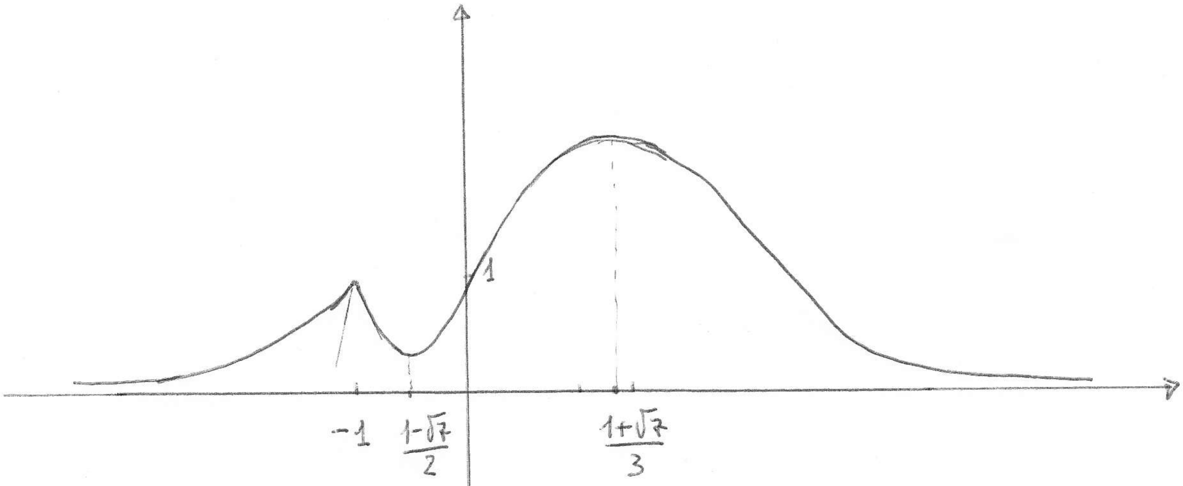
$y' > 0$ per $x < -1 \ \vee\ \frac{1-\sqrt{7}}{3} < x < \frac{-1+\sqrt{7}}{3}$

f crescente



M $\begin{cases} x = -1 \\ y = 1 \end{cases}$ M $\begin{cases} x = \frac{1+\sqrt{7}}{3} \\ y = f\left(\frac{1+\sqrt{7}}{3}\right) \end{cases}$ m $\begin{cases} x = \frac{1-\sqrt{7}}{3} \\ y = f\left(\frac{1-\sqrt{7}}{3}\right) \end{cases}$

Si omette lo studio di y''.



4. ID: $\begin{cases} \dfrac{x^4-2x^2-3}{x^2-4} \geq 0 \\ x \neq 0 \end{cases}$

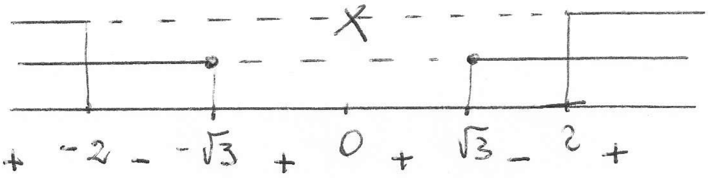
$N \geq 0 \quad x^4-2x^2-3 \geq 0 \quad x^2=t$

$t^2-2t-3 \geq 0 \quad t \leq -1 \vee t \geq 3$

$x^2 \leq -1 \ \emptyset \ \vee \ x^2 \geq 3 \iff$

$D>0 \quad x^2-4>0 \quad x<-2 \vee x>2$

$x \leq -\sqrt{3} \ \vee \ x \geq \sqrt{3}$



$ID =]-\infty,-2[\cup [-\sqrt{3},0[\cup]0,\sqrt{3}] \cup]2,+\infty[$

Poiché $f(-x) = -f(x)$ basta studiare i limiti per $x>0$

$\lim_{x\to 0^\pm} \sqrt{\dfrac{x^4-2x^2-3}{x^2-4}} \operatorname{sen}\dfrac{1}{x} = \sqrt{\dfrac{3}{4}} \cdot \lim_{y\to\pm\infty} \operatorname{sen} y \ \nexists$ perché seno è una funzione periodica

$\lim_{x\to 2^+} \sqrt{\dfrac{x^4-2x^2-3}{x^2-4}} \operatorname{sen}\dfrac{1}{x} = \sqrt{\dfrac{16-8-3}{0^+}} \cdot \operatorname{sen}\dfrac{1}{2} = +\infty \qquad x=2$ A.V.

$\lim_{x\to+\infty} \sqrt{\dfrac{x^4-2x^2-3}{x^2-4}} \operatorname{sen}\dfrac{1}{x} = \lim_{x\to+\infty} \dfrac{\cancel{x^2}}{\cancel{x}} \sqrt{\dfrac{1-2/x^2-3/x^4}{1-4/x^2}} \cdot \operatorname{sen}\dfrac{1}{x}$

$= \lim_{x\to+\infty} \sqrt{\dfrac{1-2/x^2-3/x^4}{1-4/x^2}} \cdot \dfrac{\operatorname{sen}\frac{1}{x}}{\frac{1}{x}} = 1 \cdot \lim_{y\to 0} \dfrac{\operatorname{sen} y}{y} = 1$

$y=1$ A. orizz. a $+\infty$

Poiché la funzione è dispari, segue che

$\lim_{x\to -2^-} \sqrt{\dfrac{x^4-2x^2-3}{x^2-4}} \operatorname{sen}\dfrac{1}{x} = -\infty \qquad x=-2$ A.V.

$$\lim_{x\to-\infty} \sqrt{\frac{x^4-2x^2-3}{x^2-4}}\ \mathrm{sen}\,\frac{1}{x} = -1 \qquad y=-1 \qquad \text{A. orizz. a } -\infty$$

5. i) Posto $\mathrm{sen}\,x = t$ si ha $\cos x\, dx = dt$ e quindi

$$\int \cos x \cdot \lg \frac{1+\mathrm{sen}\,x+\cos^2 x}{\mathrm{sen}\,x}\, dx = \int \lg \frac{1+t+1-t^2}{t}\, dt =$$

$$\int (Dt) \lg \frac{-t^2+t+2}{t}\, dt \overset{p.p.}{=} t \lg \frac{-t^2+t+2}{t} - \int t \frac{t}{-t^2+t+2}\, \frac{(-2t+1)t+t^2-t-2}{\cancel{t^2}}\, dt$$

$$= t \lg \frac{-t^2+t+2}{t} - \int \frac{t^2+2}{t^2-t-2}\, dt.$$

Si ha

$$\int \frac{t^2+2}{t^2-t-2}\, dt = \int \left(1 + \frac{t+4}{(t+1)(t-2)}\right) dt$$

$$\begin{array}{r|l} t^2 \qquad +2 & t^2-t+2 \\ \hline t^2-t-2 & 1 \\ \hline // \quad t+4 & \end{array}$$

$$= t + \int \frac{t+4}{(t+1)(t-2)}\, dt = \qquad t^2-t-2=0 \iff (t+1)(t-2)$$

$$= t + \int \left(\frac{-1}{t+1} + \frac{2}{t-2}\right) dt = t - \lg|t+1| + 2 \lg|t-2| + c.$$

(Infatti $\dfrac{t+4}{(t+1)(t-2)} = \dfrac{A}{t+1} + \dfrac{B}{t-2} = \dfrac{(A+B)t - 2A+B}{(t+1)(t-2)} \iff$

$$\begin{cases} A+B=1 \\ -2A+B=4 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=-A+1=2 \end{cases}$$
$$3A \quad = -3$$
)

Conclusione:

$$I = \mathrm{sen}\,x \lg \frac{-\mathrm{sen}^2 x+\mathrm{sen}\,x+2}{\mathrm{sen}\,x} - \mathrm{sen}\,x + \lg(1+\mathrm{sen}\,x) - 2\lg(2-\mathrm{sen}\,x) + c.$$

ii) Poiché $\dfrac{1+\mathrm{sen}\,x+\cos^2 x}{\mathrm{sen}\,x} > 0 \iff \mathrm{sen}\,x > 0 \iff D = \bigcup_{k\in\mathbb{Z}}]2k\pi, \pi+2k\pi[$

h non è integrabile in $]3,+\infty$ perché non è ivi definita.

Invece h è integ. in s.i. in $]0,\frac{\pi}{2}]$ perché continua e

$$\lim_{x\to 0} \Big[\underbrace{\cos x \lg(1+\mathrm{sen}\,x+\cos^2 x)}_{\to \lg 2} - \underbrace{\cos x}_{\to 1}\, \underbrace{\lg \mathrm{sen}\,x}_{\to -\infty}\Big] = +\infty$$ con ordine comunque piccolo.

Infatti $\forall \alpha > 0$: $\lim_{x\to 0^+} \frac{\lg \operatorname{sen} x}{\frac{1}{x^\alpha}} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{\cos x}{\operatorname{sen} x}}{-\alpha x^{-\alpha-1}} = \lim_{x\to 0^+} \underbrace{-\frac{\cos x}{\alpha}}_{\downarrow\, -\frac{1}{\alpha}} \cdot \underbrace{x^\alpha}_{\downarrow\, 0} \cdot \underbrace{\frac{x}{\operatorname{sen} x}}_{\downarrow\, 1} = 0$

Analogamente:

h è integ. in s. i. in $\left[\frac{\pi}{2}, \pi\right[$ perchè continua e

$$\lim_{x\to\pi^-} \Big[\underbrace{\cos x \lg(1+\operatorname{sen} x+\cos^2 x)}_{\downarrow\, -\lg 2} - \underbrace{\cos x}_{\downarrow\, +1} \underbrace{\lg \operatorname{sen} x}_{\downarrow\, -\infty}\Big] = -\infty$$

con ordine comunque piccolo.

Infatti

$$\lim_{x\to\pi^-} \lg \operatorname{sen} x \underset{\pi - x = y}{=} \lim_{y\to 0^+} \lg(\operatorname{sen}(\pi - y)) = \lim_{y\to 0^+} \lg \operatorname{sen} y = -\infty$$

con ordine comunque piccolo per quanto visto nel caso precedente.