LEZIONE 33

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Polinomi di Taylor delle funzioni elementari

1) $f(x) = e^x$, $x_0 = 0$

$$T_m(x) = \sum_{k=0}^{m} \frac{x^k}{k!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots + \frac{x^m}{m!}$$

2) $f(x) = \sin x$, $x_0 = 0$

$$T_{2m+1}(x) = \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} + \dots + (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

3) $f(x) = \cos x$, $x_0 = 0$

$$T_{2m}(x) = \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots \pm (-1)^m \frac{x^{2m}}{(2m)!}$$

4) $f(x) = \frac{1}{1-x}$, $x_0 = 0$

$$T_m(x) = \sum_{k=0}^{m} x^k = 1 + x + \dots + x^m$$

5) $f(x) = \frac{1}{1+x}$, $x_0 = 0$

$$T_m(x) = \sum_{k=0}^{m} (-1)^k x^k = 1 - x + x^2 - x^3 + \dots + (-1)^m x^m$$

6) $f(x) = \ln(1+x)$, $x_0 = 0$

$$T_m(x) = \sum_{k=1}^{m} (-1)^{k+1} \frac{x^k}{k} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{m+1} \frac{x^m}{m}$$

7) $f(x) = \arctan x$, $x_0 = 0$

$$T_{2m+1}(x) = \sum_{k=1}^{m} (-1)^k \frac{x^{2k+1}}{2k+1}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + (-1)^m \frac{x^{2m+1}}{2m+1}$$

8) $f(x) = (1+x)^{\alpha}$, $x_0 = 0$

$$T_n(x) = \sum_{n=0}^{n} \binom{\alpha}{n} x^n \text{ dove } \binom{\alpha}{k} = \begin{cases} 0 & \text{se } k=0 \\ \dfrac{\alpha\cdot(\alpha-1)\cdot\ldots\cdot(\alpha-k+1)}{k!} & \text{se } k\geq 1 \end{cases}$$

$$= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6} x^3 + \ldots$$

ESERCIZIO

Calcolare $\displaystyle\lim_{x\to 0} \frac{\cos x \cdot \ln(1+x) - x}{2\sqrt{1+x} - \sin x - 2\cos x}$

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}x^2 + o(x^2)$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)$$

$$\sin x = x - \underbrace{\frac{x^3}{6} + o(x^3)}_{o(x^2)} = x + o(x^2)$$

$$\cos x = 1 - \frac{1}{2}x^2 + o(x^2)$$

Denominatore $(D(x))$

$$D(x) = 2\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)\right) - (x + o(x^2)) - 2\left(1 - \frac{1}{2}x^2 + o(x^2)\right)$$

$$= \cancel{2} + \cancel{x} - \frac{1}{4}x^2 + o(x^2) - \cancel{x} - o(x^2) - \cancel{2} + x^2 + o(x^2)$$

$$= x^2 - \frac{1}{4}x^2 + o(x^2)$$

$$= \frac{3}{4}x^2 + o(x^2).$$

Numeratore: $\cos x \cdot \ln(1+x) - x$

$$\cos x = 1 + o(1)$$
$$\ln(1+x) = x + o(x)$$

$$\cos x \cdot \ln(1+x) = (1+o(1))(x+o(x))$$
$$= x + \underbrace{o(1)\,x + o(x) + o(1)\,o(x)}_{o(x)}$$
$$= x + o(x)$$

$$\cos x \cdot \ln(1+x) - x = o(x)$$

Rifacciamo il conto in maniera più precisa

$$\cos x = 1 + o(x)$$
$$\ln(1+x) = x - \frac{1}{2}x^2 + o(x^2)$$
$$\cos x \cdot \ln(1+x) = (1+o(x))\left(x - \frac{1}{2}x^2 + o(x^2)\right)$$
$$= x - \frac{1}{2}x^2 + \underbrace{o(x^2) + o(x^2) + o(x^3) + o(x^3)}_{o(x^2)}$$
$$= x - \frac{1}{2}x^2 + o(x^2)$$

Numeratore $\cos x \ln(1+x) - x = -\frac{1}{2}x^2 + o(x^2)$

Conclusione

$$\lim_{x\to 0} \frac{N(x)}{D(x)} = \lim_{x\to 0} \frac{-\frac{1}{2}x^2 + o(x^2)}{\frac{3}{4}x^2 + o(x^2)}$$
$$= \lim_{x\to 0} \frac{-\frac{1}{2}\cancel{x^2}}{\frac{3}{4}\cancel{x^2}} = \frac{-\frac{1}{2}}{\frac{3}{4}}$$
$$= -\frac{1}{2}\cdot\frac{4}{3} = -\frac{2}{3}$$

Domanda: Come si trova il polinomio di Taylor di e^{-x}?

Se $x \to 0$, anche $y = -x \to 0$.

$$e^{-x} \overset{y=-x}{=} e^{y}$$

$$= 1 + y + \frac{1}{2} y^2 + \frac{1}{6} y^3 + \cdots + \frac{1}{n!} y^n + o(y^n)$$

$$= 1 - x + \frac{1}{2}(-x)^2 + \frac{1}{6}(-x)^3 + \cdots + \frac{1}{n!}(-x)^n + o((-x)^n)$$

$$= \underbrace{1 - x + \frac{1}{2} x^2 - \frac{1}{6} x^3 + \cdots + (-1)^n \frac{x^n}{n!}}_{= T_n(x)} + o(x^n)$$

ESEMPIO

$\sin 2x$ in $x_0 = 0$ all'ordine 5

$$\sin 2x \overset{y=2x}{=} \sin y$$

(se $x \to 0$, anche $y \to 0$)

$$= y - \frac{y^3}{6} + \frac{y^5}{120} + o(y^5)$$

$$= 2x - \frac{(2x)^3}{6} + \frac{(2x)^5}{120} + o(x^5)$$

$$= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} + o(x^5)$$

$$= 2x - \frac{4}{3} x^3 + \frac{4}{15} x^5 + o(x^5)$$

ESEMPIO

$\sqrt{1+3x^2}$ cerchiamo la formula di Taylor in $x_0 = 0$ e $n = 4$.

$$\sqrt{1+3x^2} \overset{y=3x^2}{=} \sqrt{1+y}$$

($y = 3x^2 \xrightarrow{x \to 0} 0$)

$$= 1 + \frac{1}{2} y - \frac{1}{8} y^2 + o(y^2)$$

$$= 1 + \frac{1}{2} 3 \cdot x^2 - \frac{1}{8}(3x^2)^2 + o((3x^2)^2)$$

$$= 1 + \frac{3}{2}x^2 - \frac{9}{8}x^4 + o(x^4).$$

$\ln(2+x)$ cerchiamo la formula di Taylor di ordine 3 in $x_0 = 0$.

$$\ln(2+x) = \ln(1+1+x) \overset{y=1+x}{=} \ln(1+y)$$

Ma $y = 1 + x \xrightarrow{x\to 0} 1$. NO

$$\ln(2+x) = \ln\left(2\left(1+\frac{x}{2}\right)\right)$$

$$= \ln 2 + \ln\left(1+\frac{x}{2}\right)$$

$$= \ln 2 + \ln(1+y) \quad \text{dove} \quad y = \frac{x}{2} \xrightarrow{x\to 0} 0$$

$$= \ln 2 + y - \frac{1}{2}y^2 + \frac{1}{3}y^3 + o(y^3)$$

$$= \ln 2 + \frac{x}{2} - \frac{1}{2}\left(\frac{x}{2}\right)^2 + \frac{1}{3}\left(\frac{x}{2}\right)^3 + o\left(\left(\frac{x}{2}\right)^3\right)$$

$$= \ln 2 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24} + o(x^3).$$

E se $x_0 \neq 0$?

ESEMPIO

$f(x) = e^x$ e $x_0 = 2$, $n = 4$.

Ci sono due metodi per calcolare $T_n(x)$.

1) Usare la definizione.

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$f(x) = e^x \Rightarrow f^{(k)}(x) = e^x$$

$$\Rightarrow f^{(k)}(x_0) = f^{(k)}(2) = e^2$$

$$T_4(x) = \sum_{n=0}^{4} \frac{e^2 (x-x_0)^n}{n!}$$

$$= e^2 + e^2(x-x_0) + \frac{e^2(x-x_0)^2}{2} + \frac{e^2(x-x_0)^3}{6} + \frac{e^2(x-x_0)^4}{24}. \qquad x_0 = 2$$

2) Sostituzione:

$$f(x) = e^x = e^{2+x-2}$$

$$= e^2 e^{x-2} \qquad y = x-2 \xrightarrow{x \to 2} 0$$

$$= e^2 e^y$$

$$= e^2 \left(1 + y + \frac{1}{2} y^2 + \frac{1}{6} y^3 + \frac{1}{24} y^4 + o(y^4)\right)$$

$$= \underbrace{e^2 \left(1 + x-2 + \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3 + \frac{1}{24}(x-2)^2\right)}_{T_4(x) \text{ in } x_0 = 2.} + o((x-2)^4)$$

ESEMPIO

$f(x) = \sqrt{x}$ cerchiamo lo sviluppo di ordine 2 in $x_0 = 4$.

1) Definizione:

$$f(x) = \sqrt{x} \qquad f(x_0) = \sqrt{4} = 2$$

$$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-\frac{1}{2}} \qquad f'(x_0) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$f''(x) = -\frac{1}{4} x^{-\frac{3}{2}} \qquad f''(x_0) = -\frac{1}{4} \frac{1}{\sqrt{4^3}} = -\frac{1}{32}$$

Quindi:

$$T_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n \qquad x_0 = 4$$

$$= f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2$$

$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

$$\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + o\left((x-4)^2\right)$$

2) $\sqrt{x} = \sqrt{4\frac{x}{4}} = 2\sqrt{\frac{x}{4}} = 2\sqrt{1+\frac{x}{4}-1}$

$$= 2\sqrt{1+\frac{x-4}{4}} \qquad y = \frac{x-4}{4} \xrightarrow{x\to 4} 0$$

$$= 2\sqrt{1+y}$$

$$= 2\left(1+\frac{1}{2}y - \frac{1}{8}y^2 + o(y^2)\right)$$

$$= 2 + y - \frac{1}{4}y^2 + o(y^2)$$

$$= 2 + \frac{x-4}{4} - \frac{1}{4}\left(\frac{x-4}{4}\right)^2 + o\left(\left(\frac{x-4}{4}\right)^2\right)$$

$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + o((x-4)^2)$$

ESERCIZIO

Calcolare $\lim_{x\to 0} \frac{x\cos(2x^2) - \ln(1-x^3) - \sin x}{x^3\left(\sqrt{1+x} - e^{\frac{x}{2}}\right)}$ $\left.\right] N(x)$ $\left.\right] D(x)$

Denominatore:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)$$

$$e^{\frac{x}{2}} \overset{y=\frac{x}{2}}{=} e^y$$

$$= 1 + y + \frac{1}{2}y^2 + o(y^2)$$

$$= 1 + \frac{x}{2} + \frac{1}{2}\left(\frac{x}{2}\right)^2 + o\left(\left(\frac{x}{2}\right)^2\right)$$

$$= 1 + \frac{x}{2} + \frac{1}{8}x^2 + o(x^2)$$

$$\frac{D(x)}{x^3} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)$$
$$- \left(1 + \frac{x}{2} + \frac{1}{8}x^2 + o(x^2)\right)$$
$$= \cancel{1} + \cancel{\frac{1}{2}x} - \frac{1}{8}x^2 \cancel{-1} - \cancel{\frac{1}{2}x} - \frac{1}{8}x^2 + o(x^2)$$
$$= -\frac{1}{4}x^2 + o(x^2)$$
$$D(x) = x^3\left(-\frac{1}{4}x^2 + o(x^2)\right)$$
$$= -\frac{1}{4}x^5 + o(x^5)$$

<u>Numeratore</u>:

$$N(x) = x\cos(2x^2) - \ln(1-x^3) - \sin x$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$$

$$\cos(2x^2) \overset{y=2x^2}{=} \cos y$$
$$= 1 - \frac{1}{2}y^2 + o(y^2)$$
$$= 1 - \frac{1}{2}(2x^2)^2 + o((2x^2)^2)$$
$$= 1 - 2x^4 + o(x^4)$$

$$x\cos(2x^2) = x\left(1 - 2x^4 + o(x^4)\right)$$
$$= x - 2x^5 + o(x^5)$$

$$\ln(1-x^3) \overset{y=-x^3}{=} \ln(1+y)$$
$$= y - \frac{1}{2}y^2 + o(y^2)$$
$$= -x^3 - \frac{1}{2}(-x^3)^2 + o((-x^3)^2)$$
$$= -x^3 \underbrace{-\frac{1}{2}x^6 + o(x^6)}_{=o(x^5)}$$

$$= -x^3 + o(x^5)$$

$$N(x) = x\cos(2x^2) - \ln(1-x^3) - \sin x$$

$$= x - 2x^5 + o(x^5) - (-x^3 + o(x^5)) - \left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right)$$

$$= \cancel{x} - 2x^5 + o(x^5) + x^3 + o(x^5) - \cancel{x} + \frac{x^3}{6} - \frac{x^5}{120} + o(x^5)$$

$$= \frac{7}{6}x^3 - 2x^5 - \frac{x^5}{120} + o(x^5)$$

$$= \frac{7}{6}x^3 + o(x^3)$$

$$\lim_{x\to 0} \frac{N(x)}{D(x)} = \lim_{x\to 0} \frac{\frac{7}{6}x^3 + o(x^3)}{-\frac{1}{4}x^5 + o(x^5)}$$

$$= \lim_{x\to 0} \frac{\frac{7}{6}\cancel{x^3}}{-\frac{1}{4}x^{\cancel{5}\,2}} = \lim_{x\to 0} \frac{\frac{7}{6}}{-\frac{1}{4}x^2} = \frac{\frac{7}{6}}{-\frac{1}{4}} \cdot \frac{1}{0^+}$$

$$= -\infty.$$

ESEMPIO

$$\lim_{x\to 0} \frac{x - \sin x + x^4}{x^3}$$

$$\sin x = x + o(x)$$

$$x - \sin x + x^4 = \cancel{x} - \cancel{x} + o(x) + x^4$$

$$= x^4 + \underline{o(x)}$$

Non è $o(x^4)$.

$$\sin x = x - \frac{1}{6}x^3 + o(x^3)$$

$$x - \sin x + x^4 = \cancel{x} - \cancel{x} + \frac{1}{6}x^3 + o(x^3) + x^4$$

$$= \frac{1}{6}x^3 + \underbrace{o(x^3) + x^4}_{o(x^3)} = \frac{1}{6}x^3 + o(x^3).$$

$$\lim_{x \to 0} \frac{\frac{1}{6}x^3 + o(x^3)}{x^3} = \frac{1}{6}$$

ESERCIZIO

$$\lim_{x \to 0} \frac{\arctan^2 x - x^2}{x^2 + 2\ln(\cos x)}$$

<u>Numeratore</u>:

$$\arctan x = x - \frac{x^3}{3} + o(x^3)$$

$$\arctan^2 x = (\arctan x)^2$$

$$= \left(x - \frac{x^3}{3} + o(x^3)\right)^2$$

$$= x^2 + \frac{x^6}{9} + o(x^6) - 2\frac{x^4}{3} + o(x^4)$$

$$+ o(x^6)$$

$$= x^2 - \frac{2}{3}x^4 + o(x^4)$$

$$N(x) = \cancel{x^2} - \frac{2}{3}x^4 + o(x^4) - \cancel{x^2}$$

$$= -\frac{2}{3}x^4 + o(x^4)$$

<u>Denominatore</u>: $x^2 + 2\ln(\cos x)$

$$\ln(\cos x) = \ln\Big(1 \underbrace{- \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)}_{y}\Big)$$

$$= \ln(1+y)$$

$$= y - \frac{1}{2}y^2 + o(y^2)$$

$$= -\frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4) - \frac{1}{2}\left(-\frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)\right)^2$$

$$+ o\left(\left(-\frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)\right)^2\right)$$

$$= -\frac{1}{2}x^2 + \frac{1}{24}x^4 + \underline{o(x^4)} - \frac{1}{2}\left(\frac{1}{4}x^4 + o(x^4)\right)$$

$$+ o\left(\frac{1}{4}x^4 + o(x^4)\right)$$

$$= -\frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{8}x^4 + o(x^4)$$

$$= -\frac{1}{2}x^2 - \frac{1}{12}x^4 + o(x^4)$$

$$D(x) = x^2 + 2\ln(\cos x)$$

$$= x^2 + 2\left(-\frac{1}{2}x^2 - \frac{1}{12}x^4 + o(x^4)\right)$$

$$= \cancel{x^2} - \cancel{x^2} - \frac{1}{6}x^4 + o(x^4)$$

$$= -\frac{1}{6}x^4 + o(x^4)$$

$$\lim_{x\to 0}\frac{N(x)}{D(x)} = \lim_{x\to 0}\frac{-\frac{2}{3}x^4 + o(x^4)}{-\frac{1}{6}x^4 + o(x^4)} = \frac{-\frac{2}{3}}{-\frac{1}{6}}$$

$$= \frac{2}{3}\cdot 6 = 4.$$

Sia $f:]a,b[\to \mathbb{R}$ e sia $x_0 \in]a,b[$ n volte derivabile in x_0. Abbiamo visto che

$$f(x) = T_n(x) + R_n(x) \quad \text{con} \quad \lim_{x\to 0}\frac{R_n(x)}{(x-x_0)^n} = 0.$$

FORMULA DI TAYLOR CON RESTO DI LAGRANGE.

Sia $f:]a,b[\to \mathbb{R}$, $n+1$ volte derivabile in $]a,b[$ e sia $x_0 \in]a,b[$. Allora $\forall x \in]a,b[\setminus \{x_0\}$ $\exists$ c_x compreso tra x e x_0.

$$R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1} \quad \text{dove } R_n(x) = f(x) - T_n(x).$$

ESEMPIO

Approssimiamo e.

$f(x) = e^x$, $x_0 = 0$

$e = f(1)$.

$f(1) = T_n(1) + R_n(1)$. Per la formula di Taylor con resto di Lagrange

$$R_n(1) = \frac{f^{(n+1)}(c)}{(n+1)!}(1-0)^n = \frac{e^c}{(n+1)!} \qquad c \in]0,1[.$$

$$0 < R_n(1) < \frac{e}{(n+1)!} \xrightarrow{n\to+\infty} 0 \quad \text{se } n \to +\infty.$$

se $n = 5$ $\quad 0 < R_n(1) < 0{,}005$

$$T_n(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{120} = 2{,}71\overline{6}$$