$$\frac{1-2|x+1|}{x^2-5x+4} \leq 0$$

Numeratore:

$$1-2|x+1| \geq 0$$

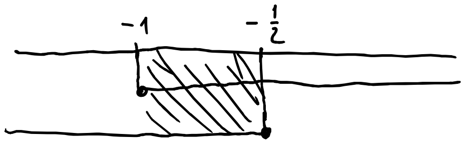
$$\begin{cases} x+1 \geq 0 \\ 1-2(x+1) \geq 0 \end{cases} \quad \vee \quad \begin{cases} x+1 < 0 \\ 1+\underbrace{2(x+1)}_{-2(-(x+1))} \geq 0 \end{cases}$$

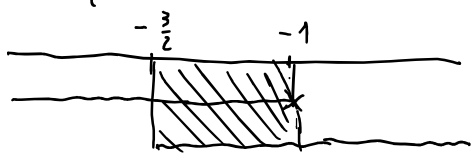
$$\begin{cases} x \geq -1 \\ 1-2x-2 \geq 0 \end{cases} \qquad \begin{cases} x < -1 \\ 1+2x+2 \geq 0 \; * \end{cases}$$

$$\begin{cases} x \geq -1 \\ -2x-1 \geq 0 \end{cases} \rightarrow -2x \geq 1 \qquad x \leq -\frac{1}{2}$$

$$* \quad 2x+3 \geq 0 \qquad 2x \geq -3 \qquad x \geq -\frac{3}{2}$$

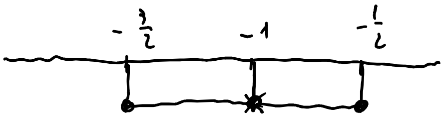
$$\begin{cases} x \geq -1 \\ x \leq -\frac{1}{2} \end{cases} \qquad \begin{cases} x < -1 \\ x \geq -\frac{3}{2} \end{cases}$$





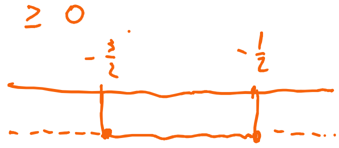
$$-1 \leq x \leq -\frac{1}{2} \qquad -\frac{3}{2} \leq x < -1$$

$$-1 \leq x \leq -\frac{1}{2} \quad \vee \quad -\frac{3}{2} \leq x < -1$$



$$-\frac{3}{2} \leq x \leq -\frac{1}{2} \;\Big]$$ Sol di "numeratore ≥ 0"

Segno del numeratore:

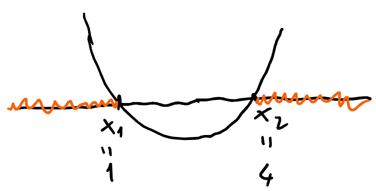


Denominatore:

$$x^2-5x+4 > 0$$

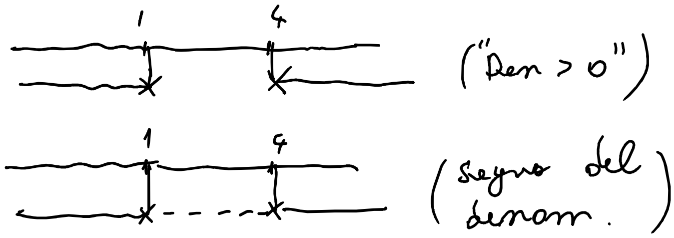
$$\Delta = (-5)^2 - 4 \cdot 1 \cdot 4 = 25-16 = 9 > 0$$

$$x_{1,2} = \frac{5 \pm \sqrt{9}}{} = \frac{5 \pm 3}{} = \nearrow 4$$

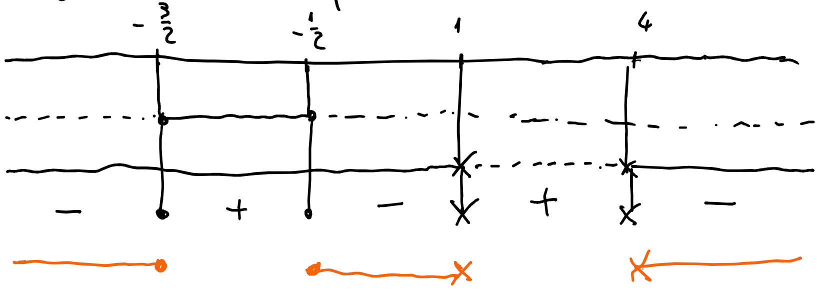


$$x_{1,2} = \frac{5 \pm \sqrt{9}}{2 \cdot 1} = \frac{5 \pm 3}{2} = \begin{cases} 4 \\ 1 \end{cases}$$

$x < 1 \quad \vee \quad x > 4$



- Segno della frazione



$\leq$

Soluzione finale: $x \leq -\frac{3}{2} \quad \vee \quad -\frac{1}{2} \leq x < 1 \quad \vee \quad x > 4.$

Limiti

Come si calcola $\lim_{x \to x_0} f(x)$?

- In molti casi, se $x_0 \in \mathbb{R}$, il limite si calcola sostituendo x_0 al posto di x (se f è continua in x_0).

$$\lim_{x \to 2} \frac{x + \frac{1}{4}}{x - 3} = \frac{2 + \frac{1}{4}}{2 - 3} = \frac{\frac{9}{4}}{-1} = -\frac{9}{4}$$

- Però possono capitare delle forme indeterminate.

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 2x + 1} = \frac{0}{0} \quad \text{(forma indeterminata)}$$

Forme indeterminate: $\frac{\pm\infty}{\pm\infty}$, $\frac{0}{0}$, $+\infty - \infty$, 0^0, $1^{\pm\infty}$, $(+\infty)^0$

Ci sono alcune situazioni standard.

1) Rapporti tra polinomi con $x \to \pm\infty$

Ci sono alcune situazioni standard.

1) Rapporti tra polinomi con $x \to \pm\infty$.

In questo caso, i termini principali sono quelli con esponente maggiore.

- $\lim_{x \to +\infty} \frac{x - x^2}{x^2 + x + 1} = \lim_{x \to +\infty} \frac{-\cancel{x^2}}{\cancel{x^2}} = \lim_{x \to +\infty} \frac{-1}{1} = -1$,

- $\lim_{x \to +\infty} \frac{3x^4 + x + 1}{11x^2 - 7x^4 + 2} = \lim_{x \to +\infty} \frac{3\cancel{x^4}}{-7\cancel{x^4}} = -\frac{3}{7}$

- $\lim_{x \to -\infty} \frac{5x^6 + x}{1 + 2x^6} = \lim_{x \to -\infty} \frac{5\cancel{x^6}}{2\cancel{x^6}} = \frac{5}{2}$

Non sempre le potenze più alte al numeratore e al denominatore hanno lo stesso esponente.

$$\lim_{x \to +\infty} \frac{x^3 - x}{1 + 2x^4} = \lim_{x \to +\infty} \frac{x^3}{2x^4} = \lim_{x \to +\infty} \frac{1}{2x} = \frac{1}{+\infty} = 0$$

$$\left(\frac{a}{\pm\infty} = 0 \right)$$

- $\lim_{x \to +\infty} \frac{1 - x^3}{x^2 + 4} = \lim_{x \to +\infty} \frac{-x^3}{x^2} = \lim_{x \to +\infty} -x = -\infty$.

- $\lim_{x \to -\infty} \frac{1 - x^3}{x^2 + 4} = \lim_{x \to -\infty} \frac{-x^3}{x^2} = \lim_{x \to -\infty} -x = -(-\infty) = +\infty$.

- $\lim_{x \to -\infty} \frac{2x^5 - 3}{3x - 4x^3 + 1} = \lim_{x \to -\infty} \frac{2x^5}{-4x^3} = \lim_{x \to -\infty} -\frac{1}{2}x^2$

$$= -\frac{1}{2}(-\infty)^2 = -\infty$$

Ricordare

1) $\frac{a}{\pm\infty} = 0 \quad \forall a \in \mathbb{R}$.

2) $(+\infty)^{\alpha} = +\infty \quad \forall \alpha \in \mathbb{R},\ \alpha > 0$

3) $(-\infty)^{\alpha} = \begin{cases} +\infty & \text{se } \alpha \in \mathbb{N},\ \alpha \text{ pari} \\ -\infty & \text{se } \alpha \in \mathbb{N},\ \alpha \text{ dispari.} \end{cases}$

---

2) limiti di rapporti tra somme di potenze con $x \to 0$.

L'idea è che i termini principali sono quelli con esponente più basso.

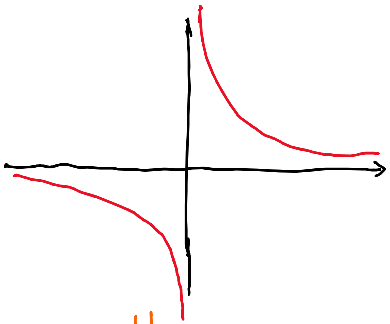
$$\lim_{x \to 0} \frac{x}{2x - x^2} = \lim_{x \to 0} \frac{\cancel{x}}{2\cancel{x}} = \frac{1}{2}$$

$$\lim_{x \to 0} \frac{x^5 - x^2 + 7x^8}{5x^2 + 3x} = \lim_{x \to 0} \frac{-x^2}{3x} = \lim_{x \to 0} -\frac{x}{3}$$

$$= -\frac{0}{3} = 0$$

Attenzione quando rimane x al denominatore:

- $\lim_{x \to 0} \frac{1}{x}$



si dice che $\nexists \lim_{x \to 0} \frac{1}{x}$ perché

$\lim_{x \to 0^+} \frac{1}{x} = +\infty$ e $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.

- $\lim_{x \to 0} \frac{1}{x^2}$

$\lim_{x \to 0} \frac{1}{x^2} = \frac{1}{0^+} = +\infty$.

- $\lim_{x \to 0} \frac{1}{x^3} \quad \nexists$ perché $\lim_{x \to 0^+} \frac{1}{x^3} = \frac{1}{0^+} = +\infty \quad \lim_{x \to 0^-} \frac{1}{x^3} = \frac{1}{0^-} = -\infty$.

- $\lim_{x \to 0} \frac{1}{x^4} = \frac{1}{0^+} = +\infty$

Ricordare

- Se $n \in \mathbb{N}\setminus\{0\}$ è pari: $\lim_{x\to 0} \frac{1}{x^n} = +\infty$
- Se $n \in \mathbb{N}$ è dispari: $\lim_{x\to 0} \frac{1}{x^n} \nexists$ perché $\lim_{x\to 0^+} \frac{1}{x^n} = +\infty$
  $\lim_{x\to 0^-} \frac{1}{x^n} = -\infty$.

$$\lim_{x\to 0} \frac{x^4 - 3x^6}{x^8 - 5x^6} = \lim_{x\to 0} \frac{x^4}{-5x^6} = \lim_{x\to 0} -\frac{1}{5x^2}$$
$$= -\left(\frac{1}{0^+}\right) = -\infty.$$

Limiti con i polinomi di Taylor

Per ogni funzione "abbastanza regolare" esiste un polinomio di grado $\leq n$ che approssima la funzione per $x \to 0$ meglio di qualsiasi altro polinomio di grado $\leq n$.

- $f(x) = e^x$

$$T_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots + \frac{1}{n!}x^n.$$

In il limite per $x \to 0$ si può sostituire $f(x)$ con $T_n(x)$ purché si tenga traccia dell'errore commesso.

Si scrive che $f(x) = T_n(x) + \underbrace{o(x^n)}_{\text{quantità trascurabile rispetto a } x^n}$.

ESEMPIO

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$$

Approssimo e^x con T_2

$$T_2(x) = 1 + x + \frac{1}{2}x^2$$

$$T_2(x) = 1 + x + \frac{1}{2}x^2$$

$$e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$$

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{\cancel{1} + \cancel{x} + \frac{1}{2}x^2 + o(x^2) - \cancel{1} - \cancel{x}}{x^2}$$

trascurabile rispetto a x^2

$$= \lim_{x\to 0} \frac{\frac{1}{2}x^2 + o(x^2)}{x^2}$$

$$= \lim_{x\to 0} \frac{\frac{1}{2}x^2}{x^2} = \frac{1}{2}$$

E se usassimo T_1?

$$e^x = 1 + x + o(x)$$

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{o(x)}{x^2} \quad ?$$ Non riesco a concludere.

Usando T_3 o T_n con n più grande si riesce a calcolare il limite ma con più conti del necessario.

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)$$

Termine principale.

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{\cancel{1} + \cancel{x} + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3) - \cancel{1} - \cancel{x}}{x^2}$$

$$= \lim_{x\to 0} \frac{\frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)}{x^2} = \lim_{x\to 0} \frac{\frac{1}{2}x^2}{x^2} = \frac{1}{2}.$$

Attenzione: la potenza più bassa può essere dentro $o(x^n)$.

$$\cancel{1} + \cancel{x} + o(x) - \cancel{1} - \cancel{x} - 3x^2 = o(x) + x^2$$

$$= o(x).$$

Questo vuol dire che il comportamento per $x \to 0$ non è stato individuato con precisione.

Altri polinomi di Taylor

- $f(x) = \cos x$

$$T_n(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \dots$$

- $f(x) = \sin x$

$$T_n(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

- $f(x) = \dfrac{1}{1-x}$

$$T_n(x) = 1 + x + x^2 + x^3 + \dots + x^n$$

- $f(x) = \ln(1+x)$

$$T_n(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 \dots$$

- $f(x) = \arctan x$

$$T_n(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$$

- $f(x) = (1+x)^\alpha$

$$T_n(x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}x^4 + \dots$$

ESEMPIO

$$f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$$

$$T_3(x) = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{6}x^3$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{\frac{3}{8}}{6}x^3$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$

Polinomi di Taylor di composizioni di funzioni.

$$f(x) = e^{3x^2}$$

Cerchiamo il polinomio di Taylor di ordine 6.

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)$$

- $$e^{3x^2} \overset{y=3x^2}{=} e^y$$
$$= 1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + o(y^3)$$
$$= 1 + 3x^2 + \frac{1}{2}(3x^2)^2 + \frac{1}{6}(3x^2)^3 + o((3x^2)^3)$$
$$= 1 + 3x^2 + \frac{9}{2}x^4 + \frac{27}{6}x^6 + o(27x^6)$$

$$= 1 + 3x^2 + \frac{9}{2}x^4 + \frac{9}{2}x^6 + o(x^6)$$

- $$2\cos x = 2\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)\right)$$
$$= 2 - x^2 + \frac{1}{12}x^4 + o(x^4)$$

- $$\cos(2x) \overset{y=2x}{=} \cos y$$
$$= 1 - \frac{1}{2}y^2 + \frac{1}{24}y^4 + o(y^4)$$
$$= 1 - \frac{1}{2}(2x)^2 + \frac{1}{24}(2x)^4 + o((2x)^4)$$
$$= 1 - \frac{1}{2}4x^2 + \frac{1}{24}\cdot 16x^4 + o(x^4)$$
$$= 1 - 2x^2 + \frac{2}{3}x^4 + o(x^4)$$

ESERCIZIO

$$\lim_{x \to 0} \frac{e^{-\frac{x^2}{2}} - \cos x}{x^2 - \ln(1+x^2)}$$

Denominatore

$$D(x) = x^2 - \ln(1+x^2)$$

$$\ln(1+x^2) \overset{y=x^2}{=} \ln(1+y) \quad \ldots \quad y^2 \quad \ldots \quad o(y^2)$$

$D(x) = x^2 - \ln(1+x^2)$

$$\ln(1+x^2) \overset{y=x^2}{=} \ln(1+y) = y - \frac{y^2}{2} + o(y^2)$$

$$= x^2 - \frac{1}{2}(x^2)^2 + o\left((x^2)^2\right)$$

$$= x^2 - \frac{1}{2}x^4 + o(x^4)$$

$$D(x) = x^2 - \left(x^2 - \frac{1}{2}x^4 + o(x^4)\right)$$

$$= \cancel{x^2} - \cancel{x^2} + \frac{1}{2}x^4 + o(x^4)$$

$$= \frac{1}{2}x^4 + o(x^4)$$

<u>Numeratore</u>

$$N(x) = e^{-\frac{x^2}{2}} - \cos x$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$$

$$e^{-\frac{x^2}{2}} \overset{y=-\frac{x^2}{2}}{=} e^y$$

$$= 1 + y + \frac{1}{2}y^2 + o(y^2)$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{2}\left(-\frac{x^2}{2}\right)^2 + o\left(\left(-\frac{x^2}{2}\right)^2\right)$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{2}\cdot\frac{x^4}{4} + o(x^4)$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)$$

$$N(x) = \cancel{1} - \cancel{\frac{1}{2}x^2} + \frac{1}{8}x^4 + o(x^4) - \left(\cancel{1} - \cancel{\frac{1}{2}x^2} + \frac{1}{24}x^4 + o(x^4)\right)$$

$$= \frac{1}{8}x^4 - \frac{1}{24}x^4 + o(x^4)$$

$$= \frac{1}{12}x^4 + o(x^4)$$

Conclusione:

$$\lim_{x\to 0}\frac{N(x)}{D(x)} = \lim_{x\to 0}\frac{\frac{1}{12}x^4 + o(x^4)}{\frac{1}{2}x^4 + o(x^4)} = \lim_{x\to 0}\frac{\frac{1}{12}\cancel{x^4}}{\frac{1}{2}\cancel{x^4}} = \frac{1}{12}\cdot 2 = \frac{1}{6}$$

$$\sin^2 x = \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5)\right)^2$$
$$= x^2 + \frac{1}{36}x^6 + \frac{1}{(120)^2}x^{10} + o(x^{10})$$
$$- \frac{1}{3}x^4 + \frac{1}{60}x^6 + \boxed{o(x^6)} - \frac{1}{3\cdot 120}x^8 + o(x^5)$$
$$+ o(x^{10})$$
$$= x^2 + \frac{1}{36}x^6 - \frac{1}{3}x^4 + \frac{1}{60}x^6 + o(x^6)$$

$$\lim_{x\to 0} \frac{\ln(1+x^2) - x\sin x}{2\arctan^2 x + \cos(2x) - 1}$$

$$N(x) = \ln(1+x^2) - x\sin x$$

$$\ln(1+x^2) \overset{y=x^2}{=} \ln(1+y)$$
$$= y - \frac{1}{2}y^2 + \frac{1}{3}y^3 + o(y^3)$$

$$= x^2 - \frac{1}{2}(x^2)^2 + \frac{1}{3}(x^2)^3 + o((x^2)^3)$$
$$= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 + o(x^6)$$

$$x\sin x = x\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5)\right)$$
$$= x^2 - \frac{1}{6}x^4 + \frac{1}{120}x^6 + o(x^6)$$

$$N(x) = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 + o(x^6) - \left(x^2 - \frac{1}{6}x^4 + \frac{1}{120}x^6 + o(x^6)\right)$$
$$= \cancel{x^2} - \frac{1}{2}x^4 + \frac{1}{3}x^6 + o(x^6) - \cancel{x^2} + \frac{1}{6}x^4 - \frac{1}{120}x^6 + o(x^6)$$
$$= -\frac{1}{3}x^4 + \underbrace{\frac{13}{40}x^6 + o(x^6)}_{\text{Trascurabili rispetto a } x^4}$$

Trascurabili rispetto a x^4

$$= -\frac{1}{3}x^4 + o(x^4)$$

$$D(x) = 2\arctan^2 x + \cos(2x) - 1$$

$$\arctan x = x - \frac{1}{3}x^3 + o(x^3)$$

$$\arctan^2 x = \left(x - \frac{1}{3}x^3 + o(x^3)\right)^2$$

$$= x^2 + \frac{1}{9}x^6 + o(x^6) - \frac{2}{3}x^4 + o(x^4) + o(x^3))$$

$$= x^2 - \frac{2}{3}x^4 + \boxed{o(x^4)}$$

$$\cos 2x \overset{y=2x}{=} \cos y$$

$$= 1 - \frac{1}{2}y^2 + \frac{1}{24}y^4 + o(y^4)$$

$$= 1 - \frac{1}{2}(2x)^2 + \frac{1}{24}(2x)^4 + o((2x)^4)$$

$$= 1 - 2x^2 + \frac{1}{24}\cdot 16x^4 + o(x^4)$$

$$= 1 - 2x^2 + \frac{2}{3}x^4 + o(x^4)$$

$$D(x) = 2\arctan^2 x + \cos 2x - 1$$

$$= 2\left(x^2 - \frac{2}{3}x^4 + o(x^4)\right) + 1 - 2x^2 + \frac{2}{3}x^4 + o(x^4) - 1$$

$$= \cancel{2x^2} - \frac{4}{3}x^4 + o(x^4) + \cancel{1} - \cancel{2x^2} + \frac{2}{3}x^4 + o(x^4) - \cancel{1}$$

$$= -\frac{2}{3}x^4 + o(x^4)$$

Risultato

$$\lim_{x\to 0}\frac{N(x)}{D(x)} = \lim_{x\to 0}\frac{-\frac{1}{3}x^4 + o(x^4)}{-\frac{2}{3}x^4 + o(x^4)} = \lim_{x\to 0}\frac{-\frac{1}{3}\cancel{x^4}}{-\frac{2}{3}\cancel{x^4}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$$2\arctan^2 x = x^2 - \frac{2}{3}x^4 + o(x^4)$$

$$2\arctan^2 x = x^2 - \frac{1}{3}x^4 + o(x^4)$$

$$\cos 2x = 1 - \frac{1}{2}(2x)^2 + o(x^2)$$

$$= 1 - 2x^2 + o(x^2)$$

$$D(x) = 2\arctan^2 x = \cancel{2x^2} - \frac{4}{3}x^4 + o(x^2) + \cancel{1} \cancel{-2x^2} + o(x^2) - \cancel{1}$$

$$= o(x^2) - \frac{4}{3}x^4$$

$$= o(x^2)$$

Nota: non dimenticate i doppi prodotti nei quadrati

$$\left(x - \frac{1}{3}x^3 + o(x^3)\right)^2$$

$$= x^2 + \frac{1}{9}x^6 + o(x^6) \underbrace{- 2x \cdot \frac{1}{3}x^3}_{-\frac{2}{3}x^4} + \underbrace{2x\, o(x^3)}_{o(x^4)} - \underbrace{2\frac{1}{3}x^3 o(x^3)}_{o(x^6)}$$

1) $\lim\limits_{x \to 0} \dfrac{\arctan(2x) - 2\sqrt{1+2x} + 2}{\ln(1+5x) - 5\sin x}$ $\left(-\frac{2}{25}\right)$

2) $\lim\limits_{x \to 0} \dfrac{(e^{2x}-1)^2 - 4x\sin x}{x\cos x - \sin x}$ (-24)

3) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x^2} - e^{\frac{x^2}{2}}}{2\cos x - 2 + x^2}$ (-3)

Soluzioni di 1) e 2) nella prossima lezione.

3) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x^2} - e^{\frac{x^2}{2}}}{2\cos x - 2 + x^2}$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$$

$$D(x) = 2\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)\right) - 2 + x^2$$

$$= \cancel{2} - \cancel{x^2} + \frac{1}{12}x^4 + \underbrace{2\,o(x^4)}_{o(x^4)} - \cancel{2} + \cancel{x^2}$$

$$= \frac{1}{12}x^4 + o(x^4)$$

$$\sqrt{1+x^2} \overset{y=x^2}{=} \sqrt{1+y} = (1+y)^{\frac{1}{2}} \qquad (1+x)^\alpha \text{ con } \alpha = \frac{1}{2}$$

$$= 1 + \alpha y + \frac{\alpha(\alpha-1)}{2!}y^2 + o(y^2)$$

$$= 1 + \frac{1}{2}y + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}y^2 + o(y^2)$$

$$= 1 + \frac{1}{2}y - \frac{1}{8}y^2 + o(y^2)$$

$$= 1 + \frac{1}{2}x^2 - \frac{1}{8}\underbrace{x^4}_{(x^2)^2 = x^4} + o(x^4)$$

$$\boxed{\begin{array}{l} \frac{1}{2} - 1 = -\frac{1}{2} \\ \dfrac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}{2} = \dfrac{-\frac{1}{4}}{2} = -\frac{1}{8} \end{array}}$$

$$e^{\frac{x^2}{2}} \overset{y=\frac{x^2}{2}}{=} 1 + y + \frac{y^2}{2} + o(y^2)$$

$$= 1 + \frac{x^2}{2} + \frac{\left(\frac{x^2}{2}\right)^2}{2} + o\left(\left(\frac{x^2}{2}\right)^2\right)$$

$$= 1 + \frac{x^2}{2} + \frac{1}{2}\cdot\frac{x^4}{4} + o(x^4)$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{8} + o(x^4)$$

$$\boxed{\begin{array}{l} \frac{1}{2}\left(\frac{x^2}{2}\right)^2 \\ = \frac{1}{2}\,\frac{x^4}{4} = \frac{x^4}{8} \end{array}}$$

$$N(x) = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4) - \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + o(x^4)\right)$$

$$= \cancel{1} + \cancel{\frac{1}{2}x^2} - \frac{1}{8}x^4 + o(x^4) - \cancel{1} - \cancel{\frac{x^2}{2}} - \frac{x^4}{8} + o(x^4)$$

$$= -\frac{1}{4}x^4 + o(x^4) \qquad \left(-\frac{1}{8} - \frac{1}{8} = -\frac{1}{4}\right)$$

$$\lim_{x\to 0} \frac{-\frac{1}{4}x^4 + o(x^4)}{\frac{1}{12}x^4 + o(x^4)} = \lim_{x\to 0} \frac{-\frac{1}{4}x^4}{\frac{1}{12}x^4} = \frac{-\frac{1}{4}}{\frac{1}{12}} = -\frac{1}{4}\cdot 12 = -3$$