Formula di Taylor

Sia $f: I \subset \mathbb{R} \to \mathbb{R}$, I intervallo di $\mathbb{R}$, f derivabile n-volte. Sia $x_0 \in I$. Allora $\forall x \in I$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2} + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + o((x-x_0)^n)$$

polinomio di Taylor centrato in x_0 di ordine n

$o((x-x_0)^n)$ è l'errore che si compie nell'approssimazione di f con il suo polinomio di Taylor

• $$\lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^3 (e^x - \cos x)}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + o(x^{2n+1})$$

$$3! = 6$$

Nel nostro caso

$$\sin x = x - \frac{x^3}{3!} + o(x^3) = x - \frac{x^3}{6} + o(x^3)$$

Allora

$$\sin^2 x = \left(x - \frac{x^3}{6} + o(x^3) \right)^2 = x^2 + \left(\frac{x^3}{6} \right)^2 +$$

$$+ \underbrace{(o(x^3))^2}_{o(x^6)} - \underbrace{2x \cdot \frac{x^3}{\cancel{6}_3}} + 2\underbrace{x\, o(x^3)}_{o(x^4)} - \frac{2}{6} \underbrace{x^3\, o(x^3)}_{o(x^6)} =$$

$$= x^2 - \frac{1}{3}x^4 \underbrace{+\frac{x^6}{36}} + o(x^4) + o(x^6)$$

Riscriviamo il numeratore

$$x^2 - \sin^2 x = x^2 - \left(x^2 - \frac{1}{3}x^4 + \frac{x^6}{36} + o(x^4) + o(x^6)\right)$$

$$= \cancel{x^2} - \cancel{x^2} + \frac{1}{3}x^4 \underbrace{- \cancel{\frac{x^6}{36}} + o(x^4) + \cancel{o(x^6)}}$$

Passiamo a studiare il denominatore

Lo sviluppo di e^x è

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n})$$

Allora

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$

Da ciò

$$e^x - \cos x = 1 + x + \frac{x^2}{2} + o(x^2) - \left(1 - \frac{x^2}{2} + o(x^2)\right)$$

$$= \cancel{1} + x + \frac{x^2}{2} - \cancel{1} + \frac{x^2}{2} + o(x^2) =$$

$$= x + x^2 + o(x^2)$$

De 45

$$\lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^3(e^x - \cos x)} = \lim_{x\to 0} \frac{\frac{1}{3}x^4 - \cancel{\frac{x^6}{36}} \cancel{+ o(x^4)} \cancel{+ o(x^6)}}{x^3(x + \cancel{x^2} \cancel{+ o(x^2)})}$$

$$= \lim_{x\to 0} \frac{\frac{1}{3}x^4}{x^3 \cdot x} = \lim_{x\to 0} \frac{\frac{1}{3}\cancel{x^4}}{\cancel{x^4}} = \frac{1}{3}$$

$$\lim_{x\to 0} \frac{\sin(3x)\cdot \ln(1+2x) - 6x^2 + 6x^3}{x^4}$$

Dobbiamo usare la formula di Taylor

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n-1}\frac{x^n}{n} + o(x^3)$$

$\forall x > -1$ ("vicino" a 0 $1+x>0$)

Allora

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + \dots + (-1)^{n-1}\frac{(2x)^n}{n} + o(x^n)$$

In particolare

$$\ln(1+2x) = 2x - 2x^2 + \frac{8}{3}x^3 + o(x^3)$$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$ (abbiamo visto prima la formula general)

Da cui

$$\sin(3x) = 3x - \frac{(3x)^3}{6} + o(x^3) =$$

$$= 3x - \frac{\cancel{27}^{\,9}}{\cancel{6}_{\,2}}x^3 + o(x^3) =$$

$$= 3x - \frac{9}{2}x^3 + o(x^3)$$

Dobbiamo calcolare

$$\sin(3x) \cdot \ln(1+2x) =$$

$$= \left(3x - \underbrace{\frac{9}{2}x^3}_{} + o(x^3)\right)\left(2x - 2x^2 + \underbrace{\frac{8}{3}x^3}_{} + o(x^3)\right) =$$

$$= 6x^2 - 6x^3 + 8x^4 + 3x\,o(x^3) - 9x^4 + 9x^5$$

$$- \frac{\cancel{9}^{3}}{\cancel{4}_{1}} \cdot \frac{\cancel{8}^{4}}{\cancel{4}_{1}} x^6 - \frac{9}{2} x^3 o(x^3) + o(x^3) \cdot 2x +$$

$$+ o(x^3)(-2x^2) + o(x^3) \frac{8}{3} x^3 + o(x^3) \cdot o(x^3)$$

$$= 6x^2 - 6x^3 - x^4 + 9x^5 - 12x^6 +$$

$$+ o(x^4) + o(x^5) + o(x^6)$$

$$= o(x^4)$$

$$= 6x^2 - 6x^3 - x^4 + o(x^4)$$

Possiamo calcolare il limite

$$\lim_{x \to 0} \frac{\sin(3x) \cdot \ln(1+2x) - 6x^2 + 6x^3}{x^4} =$$

$$= \lim_{x \to 0} \frac{\cancel{6x^2} - \cancel{6x^3} - x^4 + o(x^4) \cancel{- 6x^2} + \cancel{6x^3}}{x^4}$$

$$= \lim_{x \to 0} \frac{-x^4}{x^4} = -1$$

Traccia del 20-01-25

$$\lim_{x\to 0} \frac{\sqrt{1+x^2} - e^{\frac{x^2}{2}}}{2\cos x - 2 + x^2}$$

In generale $\qquad \alpha \neq 0, \alpha \neq -1$

$$(1+x)^\alpha = \underbrace{\binom{\alpha}{0}x^0}_{=1} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \dots$$

$$\dots + \binom{\alpha}{n}x^n + o(x^n)$$

ove $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$

In particolare $\binom{\alpha}{0} = 1 \qquad \binom{\alpha}{1} = \alpha \qquad \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{\underbrace{2!}_{2}}$

$\alpha = \frac{1}{2}$

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}x^2 + o(x^2) =$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)$$

Nel nostro caso

$$\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4)$$

Inoltre

$$e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$$

$$e^{\frac{x^2}{2}} = 1 + \frac{x^2}{2} + \frac{1}{2}\left(\frac{x^2}{2}\right)^2 + o(x^4) =$$

$$= 1 + \frac{x^2}{2} + \frac{1}{8}x^4 + o(x^4).$$

Allora

$$\sqrt{1+x^2} - e^{\frac{x^2}{2}} = 1 + \frac{x^2}{2} - \frac{1}{8}x^4 + o(x^4)$$

$$- \left(1 + \frac{x^2}{2} + \frac{1}{8}x^4 + o(x^4)\right) =$$

$$= \cancel{1} + \cancel{\frac{x^2}{2}} \underbrace{- \frac{1}{8}x^4}_{} - \cancel{1} - \cancel{\frac{x^2}{2}} \underbrace{- \frac{1}{8}x^4}_{} + o(x^4)$$

$$= -\frac{2}{8}x^4 + o(x^4) = -\frac{1}{4}x^4 + o(x^4)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$$

Il denominatore diventa

$$2\cos x - 2 + x^2 = 2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right) - 2 + x^2$$

$$= \cancel{2} - \cancel{x^2} + \frac{x^4}{12} - \cancel{2} + \cancel{x^2} + o(x^4) = \frac{x^4}{12} + o(x^4)$$

Il limite diventa

$$\lim_{x \to 0} \frac{\sqrt{1+x^2} - e^{\frac{x^2}{2}}}{2\cos x - 2 + x^2} = \lim_{x \to 0} \frac{-\frac{1}{4}x^4 + o(x^4)}{\frac{x^4}{12} + o(x^4)} =$$

$$= \lim_{x\to 0} \frac{-\frac{1}{4}\cancel{x^4}}{\frac{1}{\cancel{12}}\cancel{x^4}} = -\frac{1}{\cancel{4}}\cdot \cancel{12}^{3} = -3$$

$$\lim_{x\to 0} \frac{x\ln(1+2x)+2x^3-2\arctan(x^2)}{\underbrace{e^{2x^4}-1}}$$

$e^{x} = 1 + x + o(x)$, quindi

$e^{2x^4} = 1 + 2x^4 + o(x^4)$. Allora

$$e^{2x^4}-1 = \cancel{1} + 2x^4 + o(x^4) - \cancel{1} = \underbrace{2x^4 + o(x^4)}$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + o(x^{2n+1})$$

In particolare

$$\arctan x = x - \frac{x^3}{3} + o(x^3)$$

Nel nostro caso:

$$\arctan(x^2) = x^2 - \frac{x^6}{3} + o(x^6) \,.$$

Passiamo al termine $\ln(1+2x)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n-1} \frac{x^n}{n} + o(x^n)$$

In particolare

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \,..$$

Allora

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + o(x^3)$$

$$= 2x - 2x^2 + \frac{8}{3}x^3 + o(x^3) .$$

Calcoliamo il limite

$$\lim_{x\to 0} \frac{x\ln(1+2x) + 2x^3 - 2\arctan x^2}{e^{2x^4}-1} =$$

$$= \lim_{x\to 0} \frac{x\left(2x - 2x^2 + \frac{8}{3}x^3 + o(x^3)\right) + 2x^3 - 2\left(x^2 - \frac{x^6}{3} + o(x^6)\right)}{2x^4 + o(x^4)} .$$

$$= \lim_{x \to 0} \frac{\cancel{2x^2} - \cancel{2x^3} + \frac{8}{3}x^4 + x\,o(x^3) + \cancel{2x^3} - \cancel{2x^2} + \frac{2}{3}x^6 + o(x^6)}{2x^4 + o(x^4)}$$

$$= \lim_{x \to 0} \frac{\frac{8}{3}x^4 + \frac{2}{3}x^6 + o(x^4) + o(x^6)}{2x^4 + o(x^4)} =$$

$o(x^4)$

$$= \lim_{x \to 0} \frac{\frac{8}{3}x^4 + o(x^4)}{2x^4 + o(x^4)} = \lim_{x \to 0} \frac{\frac{8}{3}\cancel{x^4}}{2\cancel{x^4}} = \frac{8}{6} =$$

$$= \frac{4}{3}$$

ESERCIZI

$$\lim_{x \to 0} \frac{x \ln(1+2x) - \sin(2x^2)}{(1-e^x)^2 - x^2} = -2$$

(ordine 3)

$$\lim_{x \to 0} \frac{e^{3x} - 3x + \sqrt{1+9x^2}}{x^2 + x\ln(1-x)} = -9$$

(ordine 4)