Esercitaz. sullo studio di funzione

1) $f(x) = \sqrt{x^2 - \frac{8}{x}}$

$D: \quad x^2 - \frac{8}{x} \geq 0$

$$\frac{\overbrace{x^3-8}^{I}}{\underbrace{x}_{II}} \geq 0$$

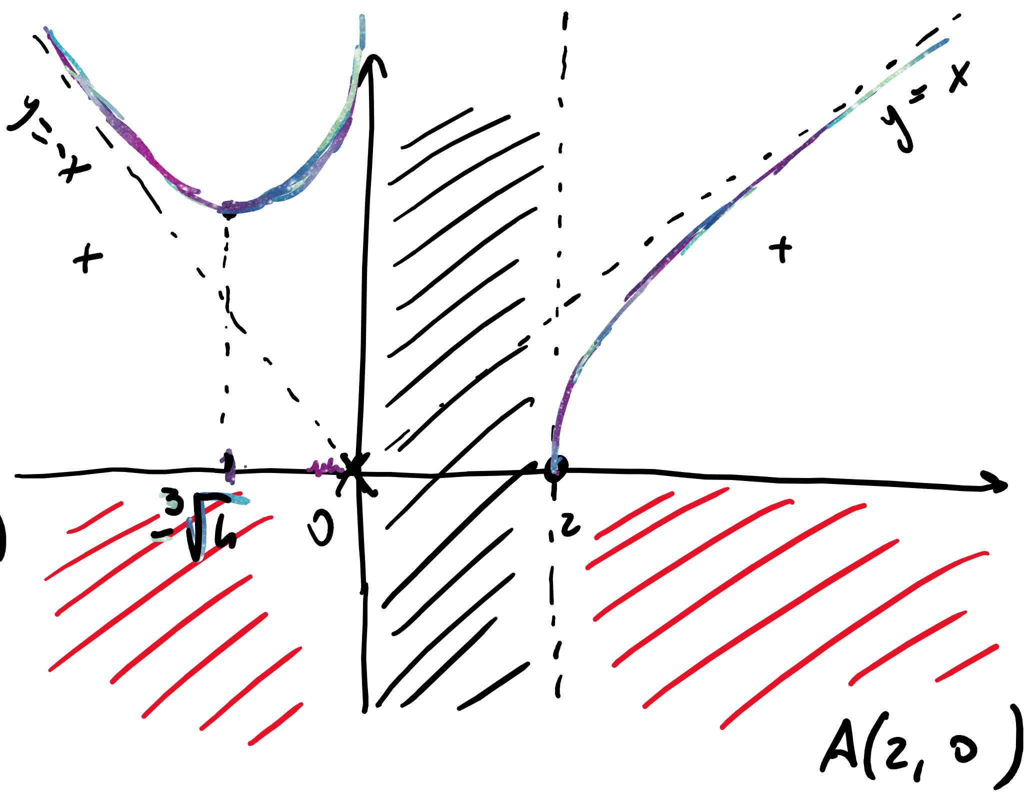
$I: \; x^3 - 8 \geq 0 \qquad x^3 \geq 8 \qquad x \geq 2$

$II: \; x > 0 \quad \text{per} \quad x > 0$

$$\begin{array}{c|ccccc} & & 0 & & 2 & \\ \hline I & - & & - & & + \\ II & - & & + & & + \\ \hline & + & & - & & + \end{array}$$

$D: \; x < 0 \;\vee\; x \geq 2$

$D =]-\infty, 0[\,\cup\, [2, +\infty[$



• Simmetrie

$f(-x) = \sqrt{x^2 + \frac{8}{x}} \neq \begin{cases} f(x) \\ -f(x) \end{cases}$

f né pari, né dispari

• Segno di f.

$\left(f(x) > 0 \qquad \sqrt{x^2 - \frac{8}{x}} > 0 \qquad \sqrt{\frac{x^3-8}{x}} > 0 \qquad \forall x \in D \setminus \{2\} \right)$

$f(x) \geq 0 \qquad \sqrt{x^2 - \frac{8}{x}} \geq 0 \qquad \forall x \in D$

$f(x) = 0 \qquad \sqrt{\frac{x^3-8}{x}} = 0 \qquad \frac{x^3-8}{x} = 0 \qquad x^3 - 8 = 0 \; (\wedge\, x \in D)$

$x^3 = 8$

$x = 2$.

$x = 8$
$x = 2$.

$A(2,0)$ intersez. asse x

- Intersez. asse y NO $0 \notin D$.

- limiti

$\lim_{x \to +\infty} \sqrt{x^2 - \frac{8}{x}} = +\infty \Rightarrow$ non c'è asintoto orizz. dx
($\frac{8}{x} \to 0$)

$\lim_{x \to -\infty} \sqrt{x^2 - \frac{8}{x}} = +\infty \Rightarrow$ non c'è asint. orizz. sx
($\frac{8}{x} \to 0$)

$\lim_{x \to 0^-} \sqrt{x^2 - \frac{8}{x}} = \left(\sqrt{+\infty}\right) = +\infty \Rightarrow$ la retta $x=0$ è asintoto verticale sx.
($\frac{8}{x} \to -\infty$)

$\left(\text{"}-\frac{8}{0^-} = +\infty\text{"} \right)$

Asintoti obliqui ?

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{\sqrt{x^2 - \frac{8}{x}}}{x} = \lim_{x \to +\infty} \frac{\sqrt{x^2\left(1 - \frac{8}{x^3}\right)}}{x}$$

$$= \lim_{x \to +\infty} \frac{\sqrt{x^2} \cdot \sqrt{1 - \frac{8}{x^3}}}{x} = \lim_{x \to +\infty} \frac{|x| \cdot \sqrt{1 - \frac{8}{x^3}}}{x}$$

$\left(\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b} \quad a, b \geq 0 \right)$

$$= \lim_{x \to +\infty} \frac{\cancel{x} \cdot \sqrt{1 - \frac{8}{x^3}}}{\cancel{x}} = 1 = m$$
($\frac{8}{x^3} \to 0$)

$$\lim_{x \to +\infty} f(x) - mx = \lim_{x \to +\infty} \sqrt{x^2 - \frac{8}{x}} - x = \quad +\infty - \infty$$

$$\lim \quad \left(\sqrt{x^2 - \frac{8}{x}} - x\right)\left(\sqrt{x^2 - \frac{8}{x}} + x\right)$$

$$\underset{\text{razionalizziamo}}{=} \lim_{x\to+\infty} \frac{\left(\sqrt{x^2-\frac{8}{x}}-x\right)\left(\sqrt{x^2-\frac{8}{x}}+x\right)}{\sqrt{x^2-\frac{8}{x}}+x}\ .$$

$$= \lim_{x\to+\infty} \frac{\cancel{x^2}-\frac{8}{x}-\cancel{x^2}}{\sqrt{x^2-\frac{8}{x}}+x} = \lim_{x\to+\infty} -\frac{8}{x\left(\sqrt{x^2-\frac{8}{x}}+x\right)}$$

$$= 0 \ (=q)$$

$\Rightarrow$ la retta $y = x$ è asintoto obliquo dx.

Analogam. per $x\to-\infty$:

$$\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\sqrt{x^2-\frac{8}{x}}}{x} = \ldots =$$

$$= \lim_{x\to-\infty} \frac{|x|\cdot\sqrt{\ldots}}{x} \underset{|x|=-x}{=}$$

$$= \lim_{x\to-\infty} \frac{-\cancel{x}\sqrt{\ldots}}{\cancel{x}} = -1 = m$$

$$\lim_{x\to-\infty} f(x) - mx = \lim_{x\to-\infty} \sqrt{x^2-\frac{8}{x}}+x \overset{+\infty-\infty}{=}$$

$$\underset{\text{razionalizz.}}{=} \lim_{x\to-\infty} \frac{\cancel{x^2}-\frac{8}{x}-\cancel{x^2}}{\underbrace{\sqrt{x^2-\frac{8}{x}}-x}_{+\infty+\infty}} = 0$$

$\Rightarrow$ la retta $y=-x$ è asintoto obliquo sx per f.

- Derivata

• Derivate

$$\rightarrow \quad f'(x) = \frac{1}{2\underbrace{\sqrt{x^2 - \frac{8}{x}}}_{>0\ \forall x \in D}} \cdot \left(2x + \frac{8}{x^2}\right) \qquad \left(D\left(\frac{1}{x}\right) = -\frac{1}{x^2} \right)$$

$$f'(x) > 0 \iff 2x + \frac{8}{x^2} > 0$$

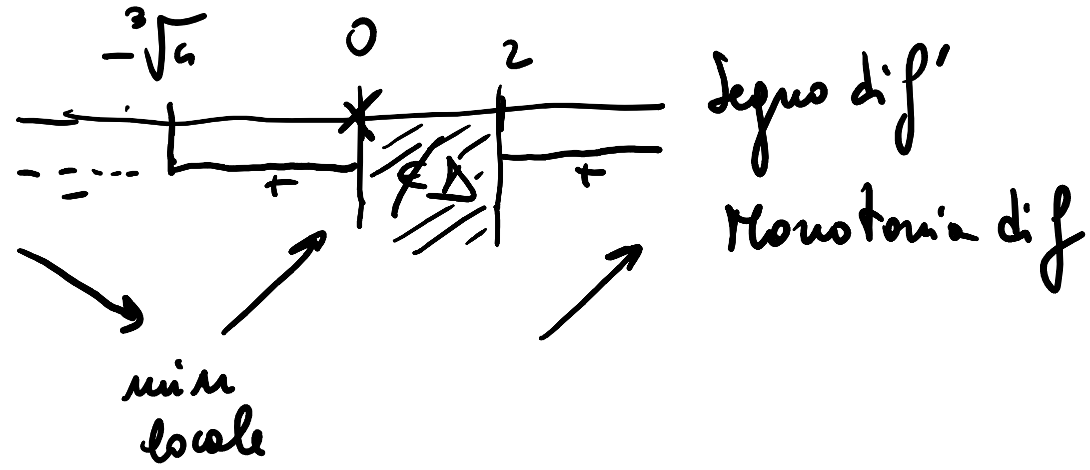
$$\cancel{2}\left(x + \frac{4}{x^2}\right) > 0$$

$$\frac{x^3+4}{\underbrace{x^2}_{>0\ \forall x \in D}} > 0 \qquad x^3 + 4 > 0$$

$$x^3 > -4$$

$$x > \sqrt[3]{-4}$$

$$x > -\sqrt[3]{4}.$$



f è decrescente in $]-\infty, -\sqrt[3]{4}]$ ed è crescente in $[-\sqrt[3]{4}, 0[$ e in $[2, +\infty[$

$x = -\sqrt[3]{4}$ è p.to di min locale

$$f(-\sqrt[3]{4}) = \ldots\ldots > 0$$

• Omettiamo lo studio di $f''(x)$.

<u>Oss.</u> f non è derivabile in $x = 2$

e $\lim_{x \to 2^+} f'(x) = \underline{+\infty} \Rightarrow x = 2$ p.to

$\lim_{x \to 2^+} f'(x) = +\infty \Rightarrow x = 2$ punto

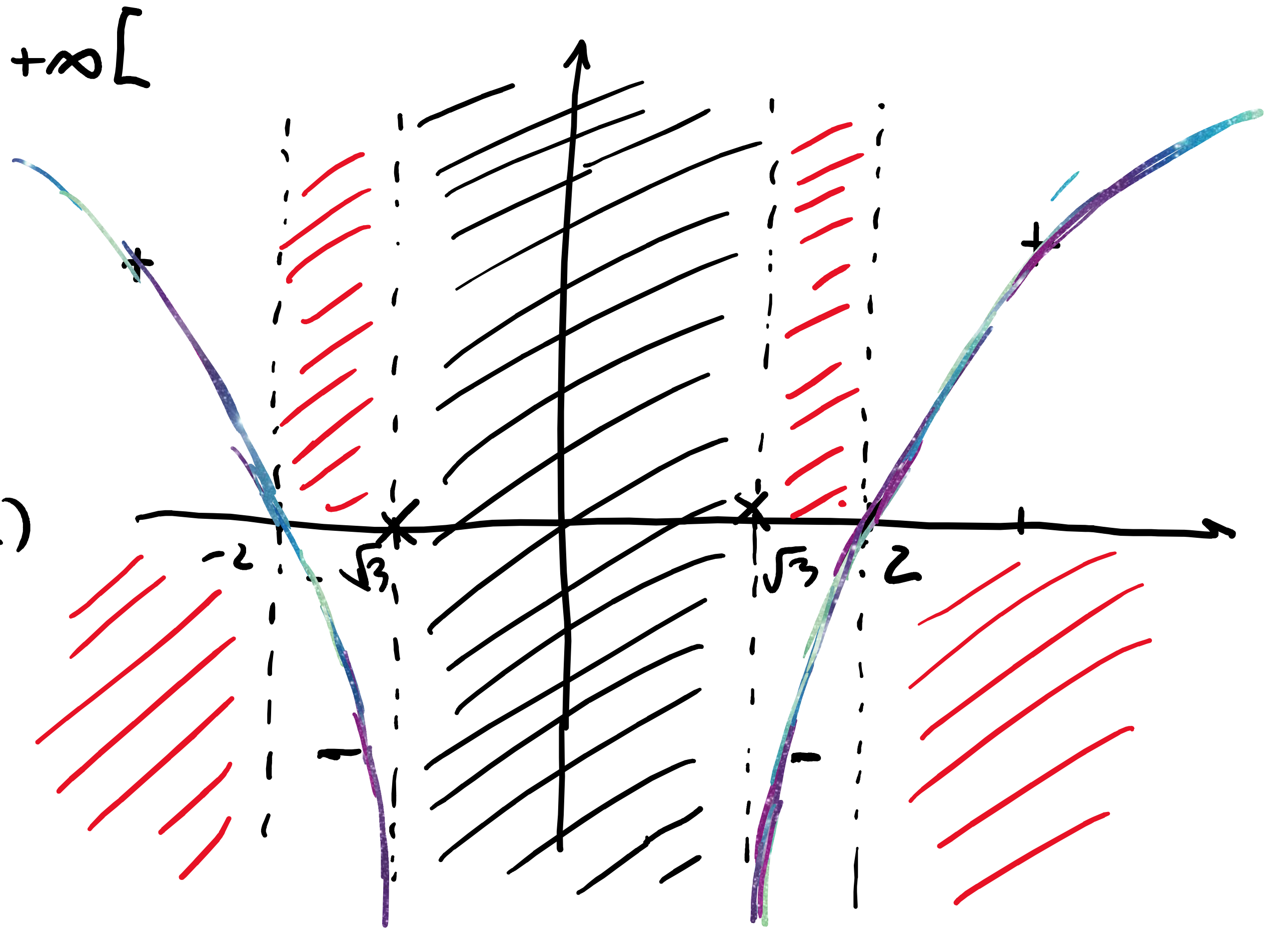
a tangente verticale.

Oss. $x = 2$ p.to di min assoluto per f.

2) $f(x) = \log(x^2 - 3)$

$D: x^2 - 3 > 0 \quad x^2 > 3 \quad x < -\sqrt{3} \vee x > \sqrt{3}$

$D =]-\infty, -\sqrt{3}[\cup]\sqrt{3}, +\infty[$



• Simmetrie

$f(-x) = \log((-x)^2 - 3)$

$= \log(x^2 - 3) = f(x)$

f è pari

• Segno di f

$\underbrace{f(x) > 0}$ $\quad \log(x^2 - 3) > 0$

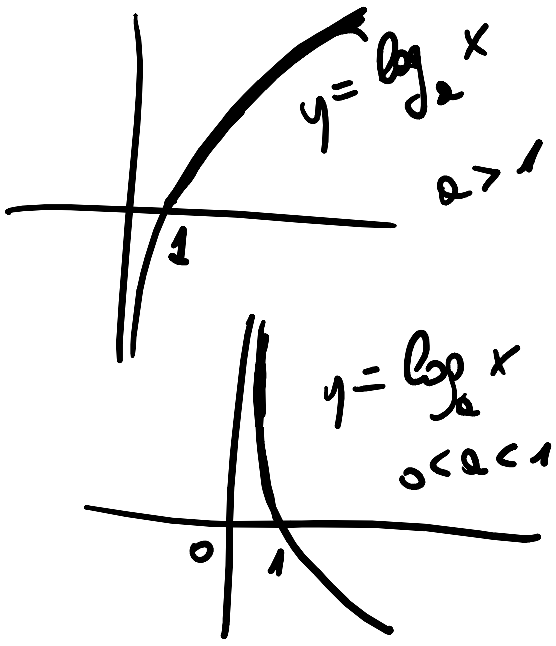
$\left(e^{\log(x^2-3)} > e^0 \right)$

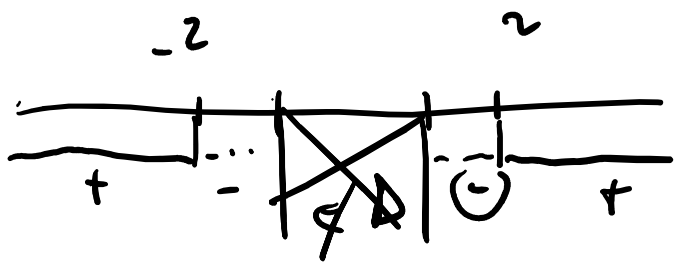
$\|$

$x^2 - 3 > 1$

$x^2 > 4$

$\underbrace{x < -2 \vee x > 2}$





$f(x) = 0 \quad \log(x^2 - 3) = 0$

$f(x) = 0 \quad \log(x^2-3) = 0$

$$x^2 - 3 = 1$$

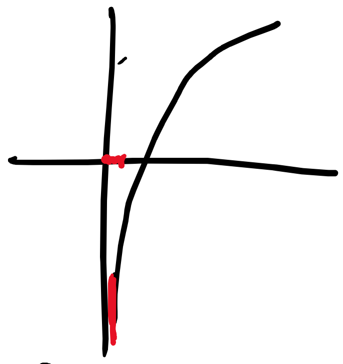
$$x^2 = 4 \qquad x = \pm 2 \quad (\text{zeri di } f)$$

$A(-2, 0)$ e $B(2, 0)$ p.ti intersez. asse x.

- $0 \notin D \Rightarrow$ No intersez. asse y.
- limiti:

$$\lim_{x \to +\infty} \log(x^2-3) = \left(\text{“}\log(+\infty)\text{”} \right) = +\infty$$

$$\lim_{x \to \sqrt{3}^+} \log(x^2-3) = \left(\text{“}\log 0^+\text{”} \right) = -\infty$$



Asint. obliquo?

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{\log(x^2-3)}{x} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{x \to +\infty} \frac{\frac{2x}{x^2-3}}{1} =$$

$$= \lim_{x \to +\infty} \frac{2x}{x^2-3} = 0$$

$\Rightarrow$ non c'è asintoto obliquo

- Derivata

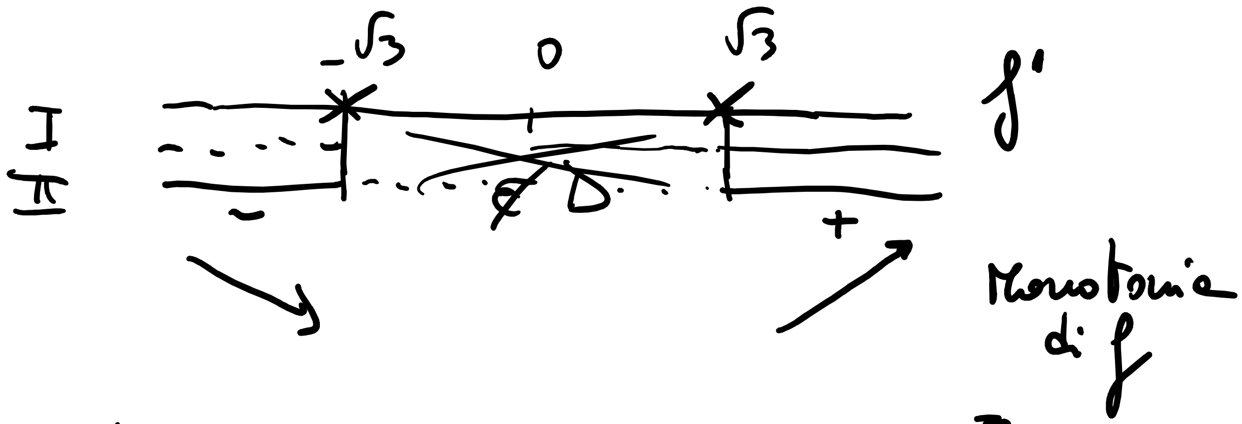
$$f'(x) = \frac{2x}{x^2-3}$$

$$f'(x) > 0 \qquad \frac{\overbrace{2x}^{I}}{\underbrace{x^2-3}_{II}} > 0$$

I: $2x > 0 \quad x > 0$

II: $x^2 - 3 > 0 \quad x^2 > 3 \qquad x < -\sqrt{3} \ \vee \ x > \sqrt{3}$

$\sqrt{3} \qquad 0 \qquad \sqrt{3}$

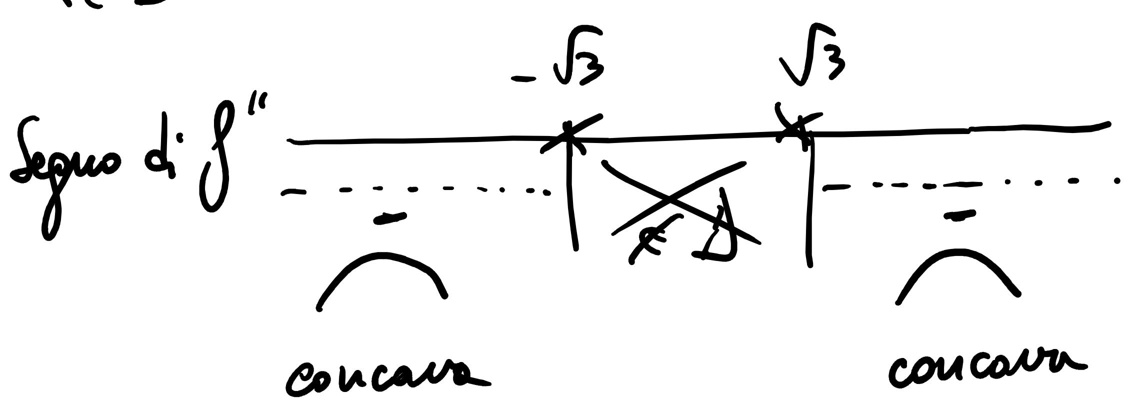


f è decrescente in $]-\infty, -\sqrt{3}[$ e crescente in $]+\sqrt{3}, +\infty[$.

- $$f''(x) = \frac{2(x^2-3) - 2x(2x)}{(x^2-3)^2} \qquad \left(f'(x) = \frac{2x}{x^2-3} \right)$$

$$= \frac{2x^2-6-4x^2}{(x^2-3)^2} = \frac{-2x^2-6}{(x^2-3)^2}$$

$$= -\frac{\overbrace{2(x^2+3)}^{>0\ \forall x \in \mathbb{R}}}{\underbrace{(x^2-3)^2}_{>0\ \forall x \in D}} \Rightarrow f''(x) < 0 \quad \forall x \in D$$



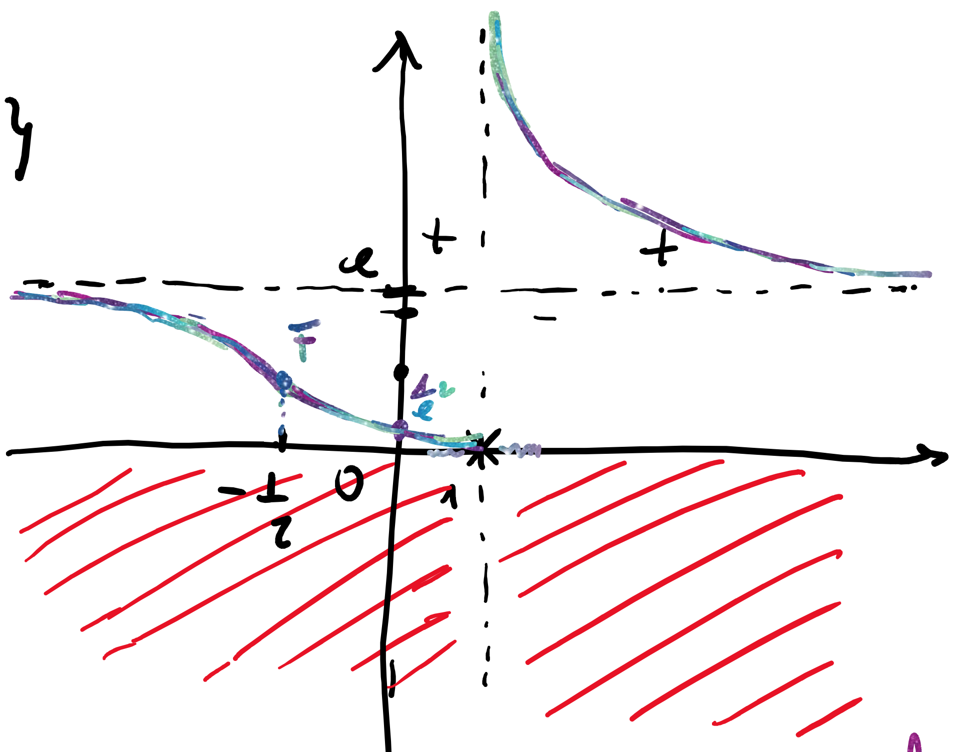
3) $f(x) = e^{\frac{x+2}{x-1}}$

D: $x-1 \neq 0$

$x \neq 1$ $\qquad D = \mathbb{R} \setminus \{1\}$

Simmetrie: NO

$$f(-x) = e^{\frac{-x+2}{-x-1}} \neq \begin{cases} f(x) \\ -f(x) \end{cases}$$



- Segno: $f(x) > 0 \qquad e^{\frac{x+2}{x-1}} > 0 \quad \forall x \in D$

$F(-\frac{1}{2}, \ldots)$ flex

• y $f(x)>0$ $\quad e^{\frac{\dots}{x-1}} > 0 \quad \forall x \in D$ ' $F(-\frac{1}{2}, \dots)$ flex.

$f(x)=0 \quad \nexists x \in \mathbb{R}$

• Interser. asse y

$$f(0) = e^{-2} = \frac{1}{e^2}$$

$A(0, \frac{1}{e^2})$ p.to interser. asse y

• Limiti

$$\lim_{x \to +\infty} e^{\frac{x+2}{x-1}} = e^1 = e$$

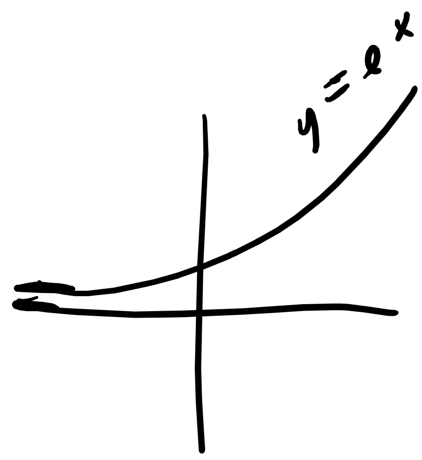
$\Rightarrow$ la retta di eq. $y = e$ è asintoto orizz. dx

$$\lim_{x \to -\infty} e^{\frac{x+2}{x-1}} = e^1 = e$$

$\Rightarrow$ $y=e$ è asintoto orizz. anche sx.

$$\lim_{x \to 1^+} e^{\frac{x+2}{x-1}} = \text{"} e^{\frac{3}{0^+}} \text{"} = \text{"} e^{+\infty} \text{"} = +\infty$$

$$\lim_{x \to 1^-} e^{\frac{x+2}{x-1}} = \text{"} e^{-\infty} \text{"} = 0$$



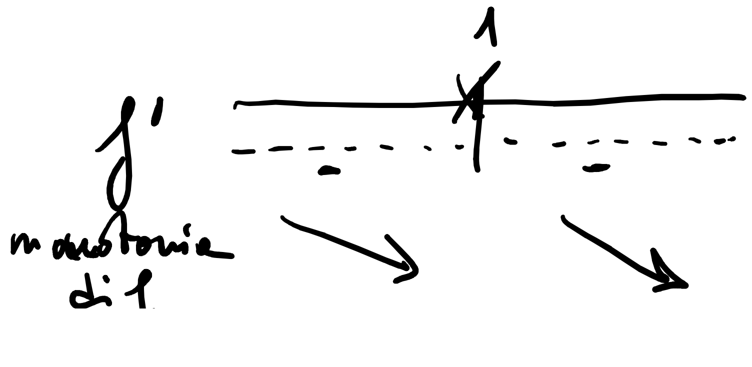
$\Rightarrow$ $x=1$ è asintoto verticale solo dx.

• Derivate

$$f'(x) = e^{\frac{x+2}{x-1}} \cdot \frac{1 \cdot (x-1) - (x+2) \cdot 1}{(x-1)^2}$$

$$= e^{\frac{x+2}{x-1}} \cdot \frac{\cancel{x} - 1 - \cancel{x} - 2}{(x-1)^2} = -\frac{3 e^{\frac{x+2}{x-1}}}{(x-1)^2}$$

$f'(x) < 0 \quad \forall x \in D$.



monotonia di f $\searrow$ $\searrow$

f è decrescente in $]-\infty, 1[$ e in $]1, +\infty[$.

- $\lim_{x\to 1^-} f'(x) = \text{"} -3\dfrac{e^{\frac{3}{0^-}}}{0^+} = -3\dfrac{e^{-\infty}}{0^+} = \dfrac{0}{0}\text{"} \rightsquigarrow 0$ conseguenza della gerarchia

$$\left(\text{Infatti } \lim_{x\to 1^-} -3\frac{e^{\frac{x+2}{x-1}}}{(x-1)^2} = \lim_{x\to 1^-} -3\frac{e^{\frac{x+2}{x-1}}}{\underbrace{(x+2)^2}_{\to 3^2}}\left(\frac{x+2}{x-1}\right)^2 \to 0\right.$$

$$\text{poiché } \lim_{x\to 1^-} e^{\frac{x+2}{x-1}}\left(\frac{x+2}{x-1}\right)^2 = \lim_{y\to-\infty} e^{y}\cdot y^2 =$$

$$\left.\underset{t=-y}{=} \lim_{t\to+\infty}\frac{t^2}{e^t} = 0 \text{ per la gerarchia degli infiniti}\right)$$

- $$f''(x) = -3\,\frac{e^{\frac{x+2}{x-1}}\cdot\dfrac{\cancel{x}-1-(\cancel{x}+2)}{\cancel{(x-1)^2}}\cdot\cancel{(x-1)^2} - e^{\frac{x+2}{x-1}}\cdot 2(x-1)}{(x-1)^4}$$

$$= \frac{-3\left[e^{\frac{x+2}{x-1}}(-3) - e^{\frac{x+2}{x-1}}\cdot 2(x-1)\right]}{(x-1)^4}$$

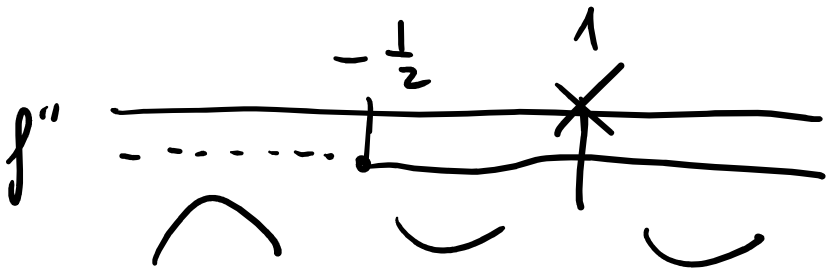
$$= \frac{-3e^{\frac{x+2}{x-1}}\left[-3-2x+2\right]}{(x-1)^4} = \frac{-3e^{\frac{x+2}{x-1}}(-2x-1)}{(x-1)^4}$$

$$= \frac{3(2x+1)\overbrace{e^{\frac{x+2}{x-1}}}^{>0\ \forall x\in D}}{\underbrace{(x-1)^4}_{>0\ \forall x\in D}}$$

$f''(x) > 0 \qquad 2x+1 > 0 \qquad x > -\frac{1}{2}$

$f''(x) > 0 \qquad 2x+1 > 0 \qquad x > -\frac{1}{2}$

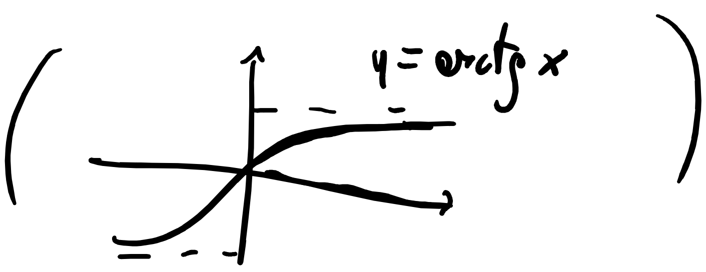
$f''(x) = 0 \qquad x = -\frac{1}{2}$



$x = -\frac{1}{2}$ p.to di flex.

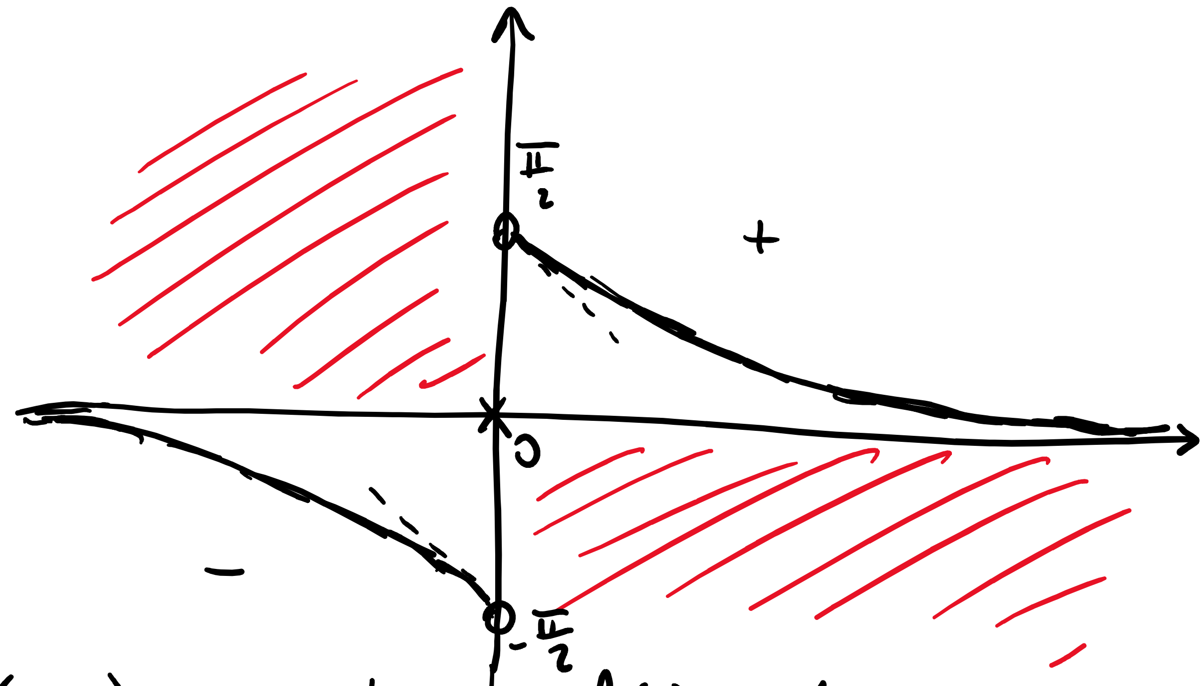
$f\left(-\frac{1}{2}\right) = \dots$

4) $f(x) = \operatorname{arctg}\left(\frac{1}{x}\right)$



D: $x \neq 0$

$D = \mathbb{R} \setminus \{0\}$



- Simmetrie

$f(-x) = \operatorname{arctg}\left(-\frac{1}{x}\right) \underset{\uparrow}{=} -\operatorname{arctg}\frac{1}{x} = -f(x)$ dispari

$y = \operatorname{arctg} x$ è dispari

- Segno

$f(x) > 0 \qquad \operatorname{arctg}\left(\frac{1}{x}\right) > 0 \qquad \frac{1}{x} > 0 \qquad x > 0$

$f(x) = 0 \qquad \operatorname{arctg}\left(\frac{1}{x}\right) = 0 \qquad \frac{1}{x} = 0$ mai

non ci sono zeri

- $0 \notin D$, no interser. asse y

- limiti

$\lim \quad \operatorname{arctg}\left(\frac{1}{x}\right) = 0$

$$\lim_{x \to +\infty} \operatorname{arctg}\Big(\underbrace{\tfrac{1}{x}}_{\to 0}\Big) = 0$$

$$\lim_{x \to -\infty} \operatorname{arctg}\Big(\underbrace{\tfrac{1}{x}}_{\to 0}\Big) = 0$$

$\Rightarrow$ la retta $y=0$ è asintoto orizz. sia dx che sx

$$\lim_{x \to 0^+} \operatorname{arctg}\Big(\underbrace{\tfrac{1}{x}}_{\to +\infty}\Big) = \frac{\pi}{2}$$

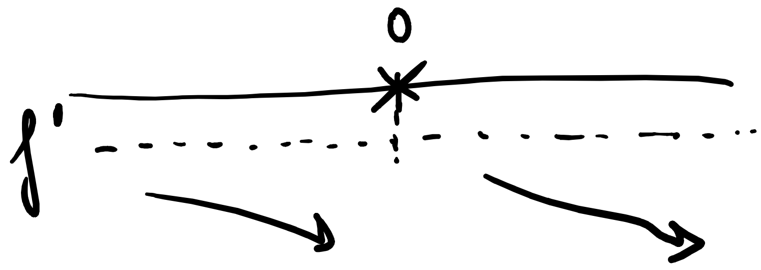
$$\lim_{x \to 0^-} \operatorname{arctg}\Big(\underbrace{\tfrac{1}{x}}_{\to -\infty}\Big) = -\frac{\pi}{2}$$

$\Rightarrow$ non c'è asintoto verticale

- Derivata

$$f'(x) = \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2+1}$$

$f'(x) > 0 \quad \nexists x \in \mathbb{R}$

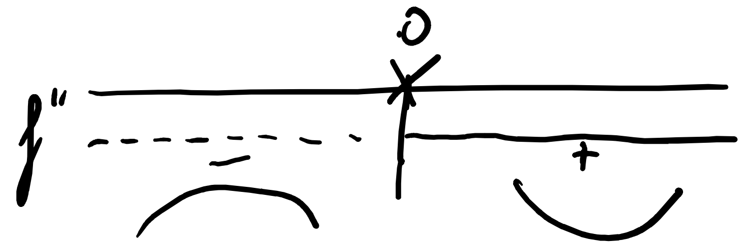


f è decrescente in $]-\infty, 0[$ e in $]0, +\infty[$

$$\left(\text{Oss. } \lim_{x \to 0^\pm} f'(x) = \lim_{x \to 0^\pm} -\frac{1}{x^2+1} = -1 \ . \right)$$

- $$f''(x) = \frac{2x}{(x^2+1)^2} \quad \text{den.} > 0 \ \forall x$$

$f''(x) > 0$ per $x > 0$



f è concava in $]-\infty, 0[$ e convessa in $]0, +\infty[$.

f è concava in $]-\infty, 0[$ e convessa in $]0, +\infty[$.