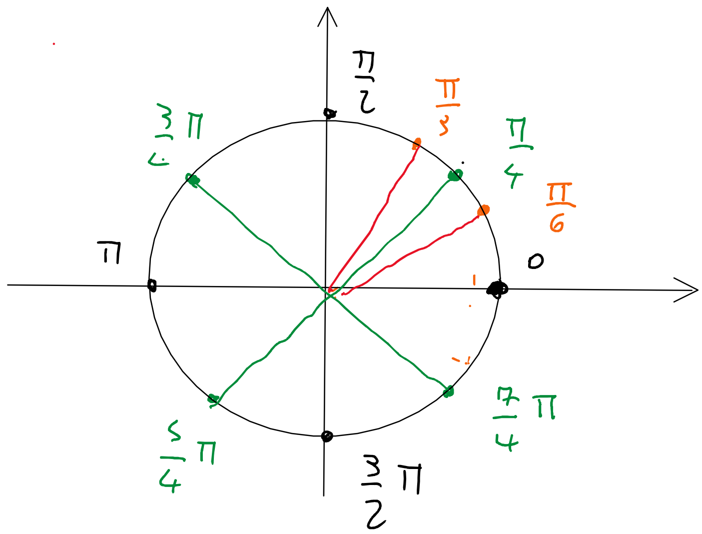
MATEMATICA - LEZIONE 14

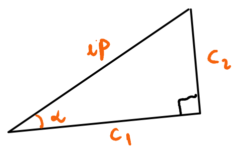
mercoledì 22 ottobre 2025 09:07

Valori noti di coseno e seno

x	$\cos x$	$\sin x$
0	1	0
$\frac{\pi}{2}$	0	1
π	-1	0
$\frac{3}{2}\pi$	0	-1
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{3}{4}\pi$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{5}{4}\pi$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
$\frac{7}{4}\pi$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$



Abbiamo detto che in un triangolo rettangolo:

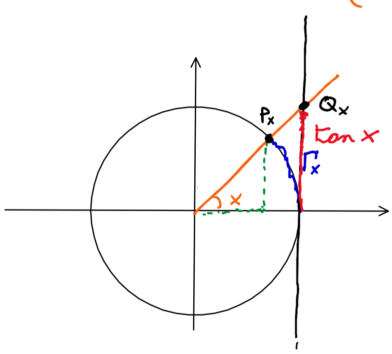


$$c_1 = ip \cdot \cos\alpha$$
$$c_2 = ip \cdot \sin\alpha$$

In particolare

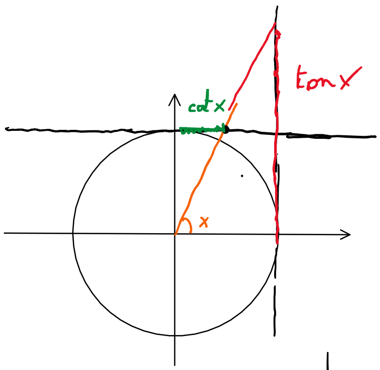
$$\frac{c_2}{c_1} = \frac{\sin\alpha}{\cos\alpha}$$ TANGENTE DI α (indicatata con: tan, tg)

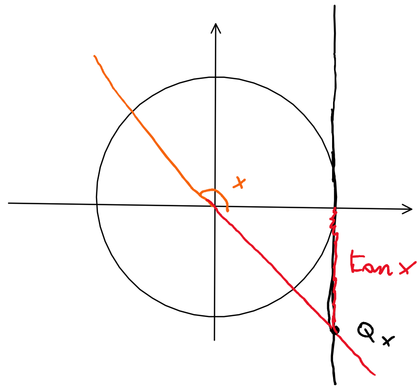
$$\frac{c_1}{c_2} = \frac{\cos\alpha}{\sin\alpha}$$ COTANGENTE DI α (indicata con: ctan, cot, ctg)

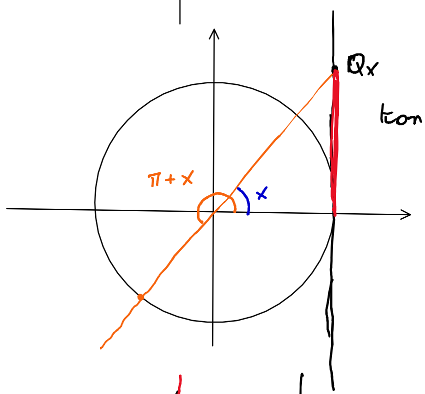


$$\frac{\sin x}{\cos x} = \frac{\tan x}{1}$$

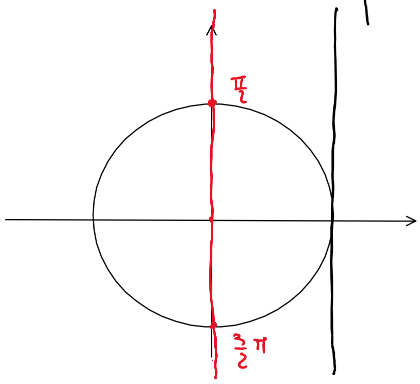
così $\tan x = \frac{\sin x}{\cos x}$







$\tan x = \tan(\pi + x)$



tan x non è definita
se $x = \frac{\pi}{2}$ o $x = \frac{3}{2}\pi$

(non definita)

x	$\cos x$	$\sin x$	$\tan x$	cot
0	1	0	0	n.d
$\frac{\pi}{2}$	0	1	n.d	0
π	-1	0	0	n.d
$\frac{3}{2}\pi$	0	-1	n.d	0
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1
$\frac{3}{4}\pi$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	-1
$\frac{5}{4}\pi$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1	1
$\frac{7}{4}\pi$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	-1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$

Ricordare:

1) $\tan x$ è definita solo se $\cos x \neq 0$
   cioè $x \neq \frac{\pi}{2} + k\pi$ con $k \in \mathbb{Z}$.

2) Mentre $|\sin x| \leq 1$, $|\cos x| \leq 1$, $\tan x$ può assumere qualsiasi valore reale.

3) $\tan x = \tan(x + k\pi) \quad \forall k \in \mathbb{Z}$.

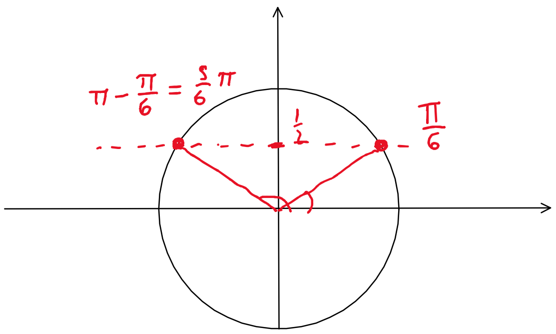
4) $\forall \alpha, \beta \in \mathbb{R}$ t.c. $\alpha, \beta \notin \left\{ \frac{\pi}{2} + k\pi \mid k \in \mathbb{Z} \right\}$ risulta che
   $\tan \alpha = \tan \beta \iff \exists\, k \in \mathbb{Z}$ t.c. $\alpha = \beta + k\pi$

ESEMPI DI EQUAZIONI ELEMENTARI CON sin E cos.

1) $\sin x = \frac{1}{2}$

Soluzione:

$x = \frac{\pi}{6} + 2k\pi$
$\vee\ x = \frac{5}{6}\pi + 2k\pi$ con $k \in \mathbb{Z}$.



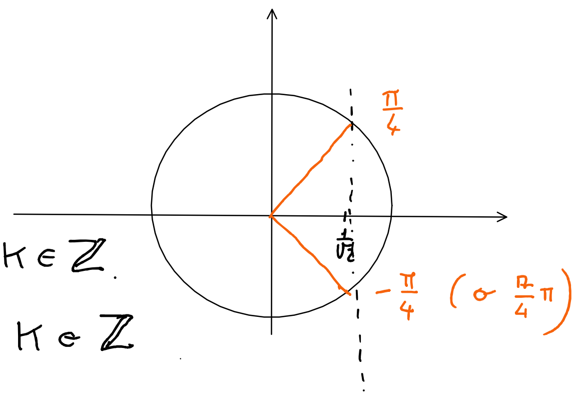
2) $\cos(\underbrace{3x}_{t}) = \frac{1}{\sqrt{2}}$

$\cos t = \frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right)$

$t = \frac{\pi}{4} + 2k\pi \ \vee\ t = -\frac{\pi}{4} + 2k\pi$ con $k \in \mathbb{Z}$.

$3x = \frac{\pi}{4} + 2k\pi \ \vee\ 3x = -\frac{\pi}{4} + 2k\pi$ con $k \in \mathbb{Z}$

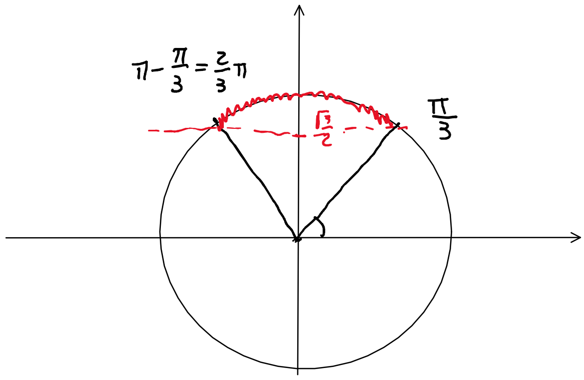
$x = \frac{\pi}{12} + \frac{2k\pi}{3} \ \vee\ x = -\frac{\pi}{12} + \frac{2k\pi}{3}$ con $k \in \mathbb{Z}$



3) $\sin(6x) \geq \frac{\sqrt{3}}{2}$

$\frac{\pi}{3} + 2k\pi \leq 6x \leq \frac{2}{3}\pi + 2k\pi$ con $k \in \mathbb{Z}$

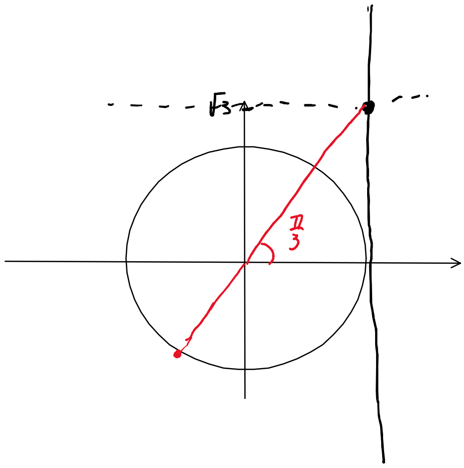
$\frac{\pi}{18} + \frac{k\pi}{3} \leq x \leq \frac{\pi}{9} + \frac{k\pi}{3}$ con $k \in \mathbb{Z}$



4) $\tan(2x) = \sqrt{3}$ $\left(= \tan\frac{\pi}{3}\right)$

$2x = \frac{\pi}{3} + k\pi$ con $k \in \mathbb{Z}$

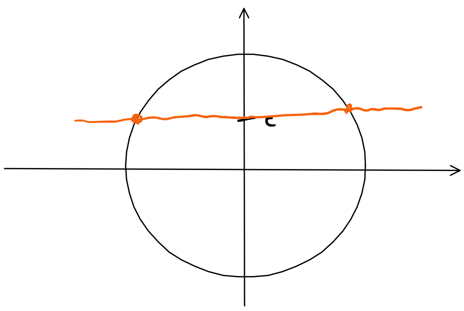
$x = \frac{\pi}{6} + \frac{k\pi}{2}$ con $k \in \mathbb{Z}$



5) $\sin(x^2) = \underbrace{\sqrt{3}}_{>1}$

impossibile!

Ricapitolando:



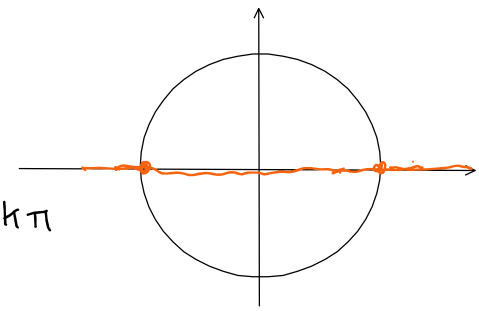
1) Per risolvere $\sin x = c$

- Se $c > 1 \vee c < -1$ l'equazione è impossibile.
- Se $c \in]-1,1[$ e conosciamo α t.c. $\sin\alpha = c$. Allora $x = \alpha + 2k\pi \vee x = \pi - \alpha + 2k\pi$
- Se $c = 1$: $x = \frac{\pi}{2} + 2k\pi$
- Se $c = -1$: $x = \frac{3}{2}\pi + 2k\pi$.

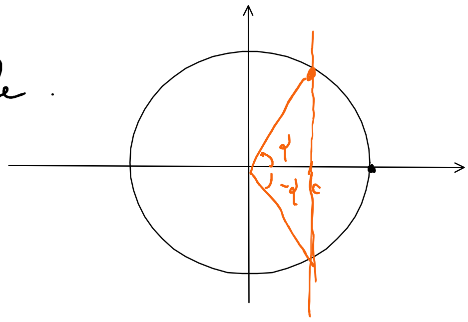
Caso particolare: $c = 0$.

$\sin x = 0 \iff x = 0 + 2k\pi \vee x = \pi + 2k\pi$

$\iff x = 0 + k\pi$ con $k \in \mathbb{Z}$.



2) $\cos x = c$.



- Se $c > 1 \ \vee \ c < -1$: impossibile.
- Se $c \in]-1,1[$ e conosciamo $\alpha \in \mathbb{R}$ t.c. $\cos \alpha = c$, allora:

  $x = \alpha + 2k\pi \quad \vee \quad x = -\alpha + 2k\pi$

  $(x = 2\pi - \alpha + 2k\pi)$

- Se $c = 1$: $x = 0 + 2k\pi$
- Se $c = -1$: $x = \pi + 2k\pi$
- Se $c = 0$: $x = \frac{\pi}{2} + k\pi$ con $k \in \mathbb{Z}$

3) $\tan x = c$

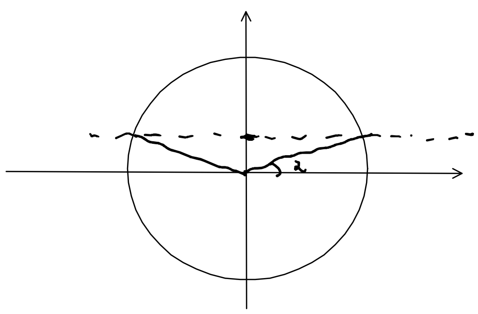
Se conosciamo un angolo t.c. $\tan \alpha = c$. Allora $x = \alpha + k\pi$ con $k \in \mathbb{Z}$.

ESEMPIO

$\sin x = \frac{1}{3}$

Se conoscessimo α t.c. $\sin \alpha = \frac{1}{3}$ allora le soluzioni sarebbero $x = \alpha + 2k\pi \quad \vee \quad x = \pi - \alpha + 2k\pi$

Problema: α non lo conosciamo esplicitamente.



Def: Sia $c \in [-1, 1]$. Si definisce ARCOSENO di c, l'unico $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ t.c. $\sin \alpha = c$.

(arcsin / arcsen / $\sin^{-1}$)

$\sin x = \frac{1}{3}$

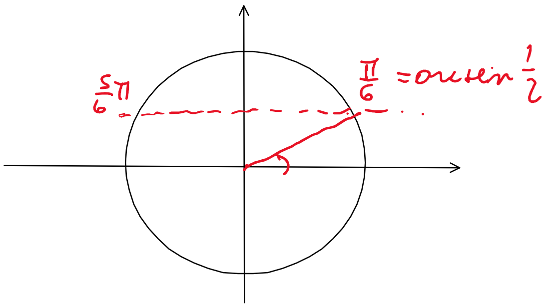
$x = \arcsin \frac{1}{3} + 2k\pi \quad \vee \quad x = \pi - \arcsin \frac{1}{3} + 2k\pi$

In generale: se $c \in]-1,1[$:

$$\sin x = c \iff x = \arcsin c + 2k\pi \quad \vee \quad x = \pi - \arcsin c + 2k\pi$$

Alcuni valori noti di arcsin:

- $\arcsin 1 = \frac{\pi}{2}$
- $\arcsin\left(\frac{1}{\sqrt{2}}\right) = \arcsin\left(\sin \frac{\pi}{4}\right) = \frac{\pi}{4}$
- $\arcsin \frac{1}{2} = \arcsin\left(\sin \frac{\pi}{6}\right) = \frac{\pi}{6}$
- $\arcsin \frac{1}{2} = \arcsin\left(\sin\left(\underbrace{\tfrac{5}{6}\pi}_{\notin [-\frac{\pi}{2}, \frac{\pi}{2}]}\right)\right) = \frac{\pi}{6}$



OSS

- $\forall x \in \mathbb{R}$, $\underline{x \in [-\frac{\pi}{2}, \frac{\pi}{2}]}$: $\arcsin(\sin x) = x$
- $\forall c \in [-1,1]$: $\sin(\arcsin c) = c$.

Def: Sia $c \in [-1,1]$: si definisce ARCOCOSENO di c l'unico $\alpha \in [0, \pi]$ t.c. $\cos \alpha = c$
(si indica con $\arccos c$)

OSS

- Sia $c \in]-1,1[$: $\cos x = c \iff x = \arccos c + 2k\pi$
$\vee \; x = -\arccos c + 2k\pi$
con $k \in \mathbb{Z}$.

- Se $x \in [0, \pi]$, allora $\arccos(\cos x) = x$.
- Sia $c \in [-1,1]$: $\cos(\arccos c) = c$.

Def: Sia $c \in \mathbb{R}$. Si definisce ARCOTANGENTE di c l'unico $\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ t.c. $\tan \alpha = c$.

oss

1) $\forall\, c \in \mathbb{R}: \quad \tan x = c \iff x = \arctan c + k\pi$ con $k \in \mathbb{Z}$
2) $\forall\, x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\quad \arctan(\tan x) = x.$
3) $\forall\, c \in \mathbb{R}: \quad \tan(\arctan c) = c.$

Ricordare

$(\arctan / \operatorname{arctg},\ \tan^{-1},\ \operatorname{tg}^{-1})$

1) $\arctan 0 = 0$
2) $\arctan 1 = \frac{\pi}{4}$
3) $\arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$
4) $\arctan \sqrt{3} = \frac{\pi}{3}$
5) $\arctan(-x) = -\arctan x$

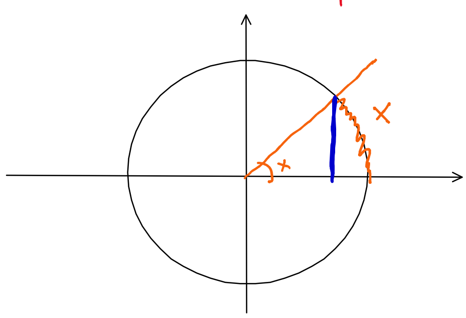
$\arctan(-1) = -\arctan 1 = -\frac{\pi}{4}.$

Seno / coseno / tangente come funzioni.

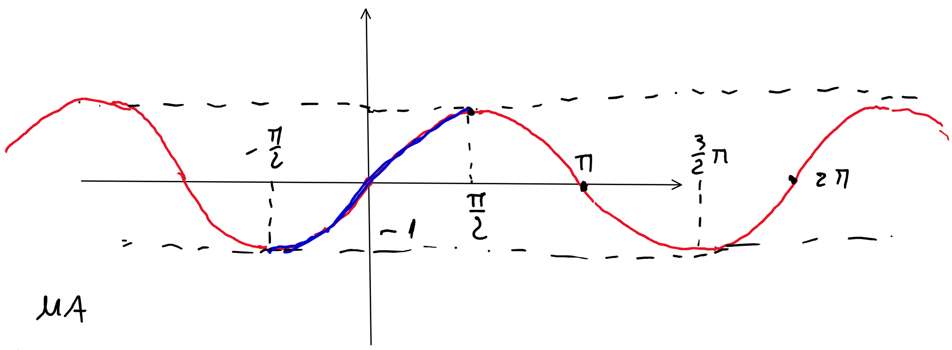
1) FUNZIONE SENO

$f(x) = \operatorname{sen} x$

$f : \mathbb{R} \longrightarrow \mathbb{R}$
$x \longmapsto \operatorname{sen} x$



- $\operatorname{Dom}(f) = \mathbb{R}$.
- $f(\mathbb{R}) = [-1, 1]$
- f è limitata.
- _ è dispari.



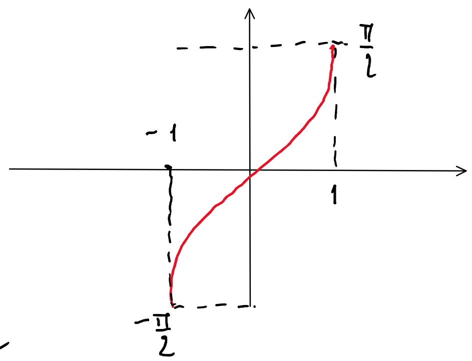
f non è invertibile MA

$g : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$
$x \longmapsto \operatorname{sen} x$

g è invertibile e g^{-1} è la funzione arcoseno.

2) FUNZIONE ARCOSENO

$$\arcsin: \begin{array}{ccc} [-1, 1] & \longrightarrow & [-\frac{\pi}{2}, \frac{\pi}{2}] \\ x & \longmapsto & \arcsin x \end{array}$$



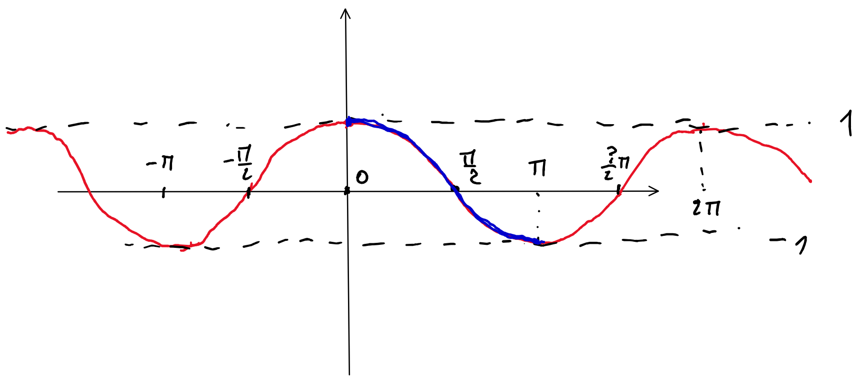
- arcsin è dispari.
- arcsin è strettamente monotona crescente.

3) FUNZIONE COSENO

$f(x) = \cos x$

$Dom(f) = \mathbb{R}$

$f(\mathbb{R}) = [-1, 1]$



f è pari.

f è limitata

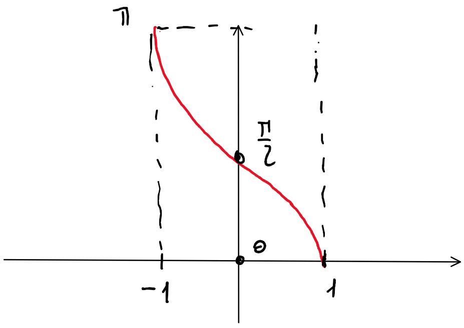
f non è invertibile MA

$$g: \begin{array}{ccc} [0, \pi] & \longrightarrow & [-1, 1] \\ x & \longmapsto & \cos x \end{array}$$ è invertibile e

g^{-1} è la funzione arccos.

4) FUNZIONE ARCOCOSENO

$$f: \begin{array}{ccc} [-1, 1] & \longrightarrow & [0, \pi] \\ x & \longmapsto & \arccos x \end{array}$$



f è strettamente monotona decrescente

5) FUNZIONE TANGENTE

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

$\mathrm{Dom}(f) = \mathbb{R} \setminus \{ \frac{\pi}{2} + k\pi \mid k \in \mathbb{Z} \}$

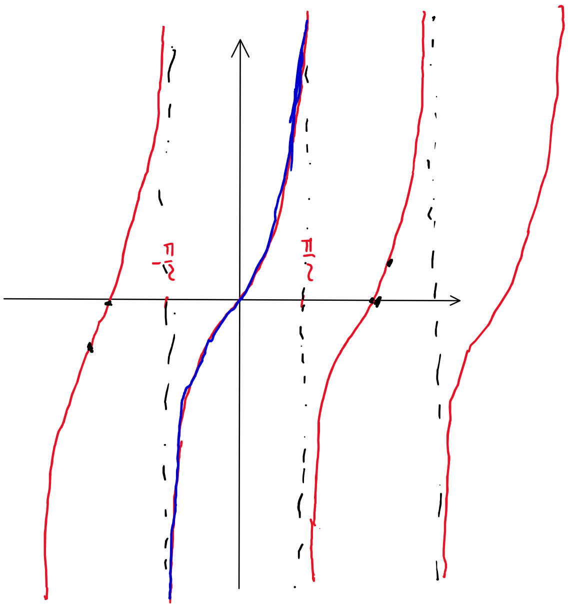
$f(\mathbb{R}) = \mathbb{R}$.

f è dispari.

f non è invertibile MA

$$g: \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\longrightarrow \mathbb{R}$$
$$x \longmapsto \tan x$$

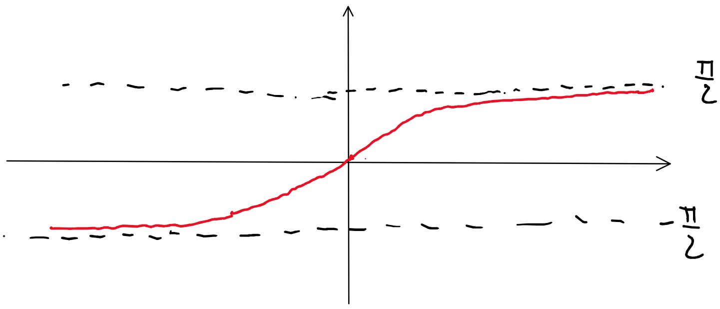
g è invertibile e la sua inversa è la funzione arcotangente.



6) ARCOTANGENTE

$$f: \mathbb{R} \longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$
$$x \longmapsto \arctan x$$

- f è limitata
- f è dispari.
- f è strettamente monotona crescente.



FORMULE PARAMETRICHE PER SIN / COS / TAN

Sia $x \in \mathbb{R}$, $x \neq \pi + 2k\pi$ con $k \in \mathbb{Z}$. Allora

1) $$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

2) $$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

3) $$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$