# Integrali:

Calcolare $\int f(x)\,dx$ significa trovare tutte le funzioni la cui derivata è $f(x)$.

ESEMPIO:

$$D\,x^2 = 2x \quad \Rightarrow \quad \int 2x = x^2 + C$$

In generale:

$$F'(x) = f(x) \iff \int f(x)\,dx = F(x) + C.$$

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## Integrali di funzioni elementari

$$\int 1\,dx = x + C$$

$$\int 3\,dx = 3x + C$$

$$\int a\,dx = ax + C$$

$$\int x\,dx = \frac{1}{2}x^2 + C$$

$$\int x^\alpha\,dx = \frac{1}{1+\alpha}x^{1+\alpha} + C \qquad (\text{se } \alpha \neq -1)$$

$$\int x^3\,dx = \frac{1}{4}x^4 + C$$

$$\int x^5\,dx = \frac{1}{6}x^6 + C$$

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{1}{-1}x^{-1} + C$$
$$= -\frac{1}{x} + C$$

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{1}{\frac{3}{2}}x^{\frac{3}{2}} + C = \frac{2}{3}x^{\frac{3}{2}} + C.$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int e^x\,dx = e^x + c$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

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Ricordare

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\int \frac{\overbrace{a}^{=f'(x)}}{\underbrace{ax+b}_{f(x)}}\,dx = \frac{1}{a}\ln|ax+b| + c$$

ESEMPI

$$\int \frac{1}{2x+1}\,dx = \frac{1}{2}\int \frac{2}{2x+1}\,dx = \frac{1}{2}\ln|2x+1| + c$$

$$\int \frac{\overbrace{1}^{=f'}}{\underbrace{x-2}_{=f}}\,dx = \ln|x-2| + C.$$

$$\int \frac{1}{x+14}\,dx = \ln|x+14| + C$$

$$\int \frac{1}{3x+14}\,dx = \frac{1}{3}\int \frac{3}{3x+14}\,dx$$

$$= \frac{1}{3}\ln|3x+14| + C.$$

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Ricordare:

$$\int \frac{f'(x)}{f(x)^2}\,dx = -\frac{1}{f(x)} + C$$

$$\left( D\left(-\frac{1}{f(x)}\right) = D\left(-f(x)^{-1}\right) = -(-1)\cdot f(x)^{-2} D f(x) = \frac{1}{f(x)^2} f'(x) = \frac{f'(x)}{f(x)^2} \right)$$

$$\int \frac{\overset{f'}{1}}{\underbrace{(x-4)}_{f(x)}{}^2}\,dx = -\frac{1}{x-4} + C$$

$$\int \frac{1}{\underbrace{(2x+1)}_{f}{}^2}\,dx \quad f'=2 \qquad = \frac{1}{2}\int \frac{\overset{f'}{2}}{\underbrace{(2x+1)}_{f}{}^2}\,dx = \frac{1}{2}\left(-\frac{1}{2x+1} + C\right)$$

$$= -\frac{1}{2}\,\frac{1}{2x+1} + \underbrace{\frac{1}{2}C}_{C_1}$$

$$= -\frac{1}{4x+2} + C_1$$

$$\int \frac{1}{(3x-1)^2}\,dx = \frac{1}{3}\int \frac{3}{(3x-1)^2}\,dx$$

$$= -\frac{1}{3}\,\frac{1}{3x-1} + C.$$

Nota 1 Attenzione ai segni

- $\int \frac{1}{1-x}\,dx = -\int \frac{-1}{1-x}\,dx = -\ln|1-x| + C$

  $\left(\text{oppure} = -\int \frac{1}{x-1}\,dx = -\ln|x-1| + C\right)$

- $\int \frac{1}{1-3x}\,dx = -\frac{1}{3}\ln|1-3x| + C$

- $\int \frac{1}{(1-x)^2}\,dx = \int \frac{1}{(x-1)^2}\,dx = -\frac{1}{x-1} + C$

  $\left(\text{oppure: } \int \frac{1}{(1-x)^2}\,dx = -\int \frac{-1}{(1-x)^2}\,dx = -\left(-\frac{1}{1-x}\right) + C = -\frac{1}{x-1} + C\right)$

$$\int \frac{1}{(3-4x)^2}\,dx = \int \frac{1}{(4x-3)^2}\,dx = -\frac{1}{4}\,\frac{1}{4x-3} + C.$$

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**Nota**

La regola che abbiamo appena visto non si può applicare $\int \frac{1}{f(x)^2}\,dx$ o $\int \frac{1}{f(x)}\,dx$

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$$\int \frac{1}{1+x^2}\,dx = \arctan x + C.$$

**Ricordare**

$$\int \frac{1}{(x-\alpha)^2+\beta^2}\,dx = \frac{1}{\beta}\arctan\left(\frac{x-\alpha}{\beta}\right) + C.$$

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ESEMPI

- $\int \frac{1}{x^2+4}\,dx \qquad (\alpha = 0,\ \beta = 2)$

$$\int \frac{1}{x^2+4}\,dx = \int \frac{1}{x^2+2^2} = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

- $\int \frac{1}{(x+\frac{1}{2})^2+\frac{7}{4}}\,dx \qquad \alpha = -\frac{1}{2},\ \beta = \frac{\sqrt{7}}{2}$

$$\parallel$$

$$\int \frac{1}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{7}}{2}\right)^2}\,dx = \frac{2}{\sqrt{7}}\arctan\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right) + C = \frac{2}{\sqrt{7}}\arctan\left(\frac{2x+1}{\sqrt{7}}\right) + C$$

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## Tre integrali da ricordare

1) $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$

2) $\int \frac{1}{(ax+b)^2}\,dx = -\frac{1}{a}\,\frac{1}{ax+b} + C$

3) $\int \frac{1}{(x-\alpha)^2+\beta^2}\,dx = \frac{1}{\beta}\arctan\left(\frac{x-\alpha}{\beta}\right) + C$

Tipico esercizio di esame:

$$\int \frac{mx+q}{ax^2+bx+c}\,dx$$

Idea: ci sono 3 casi:
Si calcola $\Delta = b^2 - 4ac$.

**1° Caso: $\Delta > 0$:** Il denominatore $ax^2+bx+c$ ha due radici $x_1$ e $x_2$ $\left(x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}\right)$
e $ax^2+bx+c = a(x-x_1)(x-x_2)$
In questo caso la frazione da integrare, si può scrivere come combinazione di $\frac{1}{x-x_1}$ e $\frac{1}{x-x_2}$.

Si cercano $A$ e $B$ t.c.

$$\frac{mx+q}{ax^2+bx+c} = \frac{A}{a(x-x_1)} + \frac{B}{x-x_2}$$

$ax^2+bx+c = a(x-x_1)(x-x_2)$

Una volta trovati $A$ e $B$:

$$\int \frac{mx+q}{ax^2+bx+c}\,dx = \int \frac{A}{a(x-x_1)}\,dx + \int \frac{B}{x-x_2}\,dx$$

$$= \frac{A}{a}\int \frac{1}{x-x_1}\,dx + B\int \frac{1}{x-x_2}\,dx$$

$$= \frac{A}{a}\ln|x-x_1| + B\ln|x-x_2| + C$$

ESEMPIO

$$\int \frac{x+1}{x^2+x-2}\,dx$$

$$\Delta = 1 - 4\cdot 1\cdot(-2) = 1+8 = 9 > 0$$

$$x_{1,2} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2} = \begin{cases} 1 \\ -2 \end{cases}$$

$$x^2+x-2 = (x-1)(x+2)$$

Cerchiamo $A$ e $B$ t.c.

$$\frac{x+1}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2} = \frac{A(x+2)+B(x-1)}{\cancel{(x-1)(x+2)}}$$

$$x+1 = A(x+2) + B(x-1)$$

$$\underline{x} + \underline{1} = \underline{Ax} + \underline{2A} + \underline{Bx} - \underline{B}$$

$$\underline{x} + \underline{1} = x\underline{(A+B)} + \underline{2A - B}$$

$$\begin{cases} 1 = A+B \\ 1 = 2A - B \end{cases} \qquad \begin{cases} B = 1-A \\ 1 = 2A - (1-A) \end{cases}$$

$$\begin{cases} B = 1-A \\ 1 = 2A - 1 + A \end{cases} \iff \begin{cases} B = 1-A \\ 2 = 3A \end{cases}$$

$$\iff \begin{cases} B = 1-A = 1-\frac{2}{3} = \frac{1}{3} \\ A = \frac{2}{3} \end{cases} \qquad A = \frac{2}{3} \quad B = \frac{1}{3}$$

$$\frac{x+1}{x^2+x-2} = \frac{\frac{2}{3}}{x-1} + \frac{\frac{1}{3}}{x+2}$$

$$\int \frac{x+1}{x^2+x-2}\,dx = \frac{2}{3}\int \frac{1}{x-1}\,dx + \frac{1}{3}\int \frac{1}{x+2}\,dx$$

$$= \frac{2}{3}\ln|x-1| + \frac{1}{3}\ln|x+2| + C$$

ESEMPIO 2

$$\int \frac{2x-1}{3x^2-5x-2}\,dx$$

$$\Delta = 25 + 4\cdot 3\cdot 2 = 49 > 0$$

$$x_{1,2} = \frac{5 \pm 7}{6} = \begin{cases} 2 \\ -\frac{1}{3} \end{cases}$$

$$3x^2 - 5x - 2 = 3(x-2)\left(x+\frac{1}{3}\right)$$
$$= (x-2)(3x+1)$$

Cerco A e B t.c.

$$\frac{2x-1}{3x^2-5x-2} = \frac{A}{x-2} + \frac{B}{3x+1}$$
$$= \frac{A(3x+1) + B(x-2)}{(x-2)(3x+1)}$$

$$2x - 1 = A(3x+1) + B(x-2)$$
$$= 3Ax + A + Bx - 2B$$

$$\begin{cases} 3A + B = 2 \\ A - 2B = -1 \end{cases}$$

$$B = 2 - 3A$$
$$A - 2(2 - 3A) = -1$$
$$A - 4 + 6A = -1$$
$$7A = 3$$
$$A = \frac{3}{7}$$

$$B = 2 - \frac{9}{7} = \frac{5}{7}$$

$$A = \frac{3}{7} \qquad B = \frac{5}{7}$$

$$\int \frac{2x-1}{3x^2-5x-2}\,dx = \frac{3}{7}\int \frac{1}{x-2}\,dx + \frac{5}{7}\int \frac{1}{3x+1}\,dx$$
$$= \frac{3}{7}\ln|x-2| + \frac{5}{7}\cdot\frac{1}{3}\ln|3x+1| + C$$
$$= \frac{3}{7}\ln|x-2| + \frac{5}{21}\ln|3x+1| + C$$

ESEMPIO 3:

$$\int \frac{3x-5}{x^2-x-2}\,dx$$

$$\Delta = 1 + 8 = 9$$

$$x_{1,2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases} \qquad x^2 - x - 2 = (x-2)(x+1)$$

$$\frac{3x-5}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$= \frac{A(x+1) + B(x-2)}{(x-2)(x+1)}$$

$$3x-5 = Ax + A + Bx - 2B$$

$$\begin{cases} A+B=3 \\ A-2B=-5 \end{cases} \longrightarrow A = 3-B$$

$$3 - B - 2B = -5$$

$$-3B = -8$$

$$B = \frac{8}{3}$$

$$A = 3 - \frac{8}{3} = +\frac{1}{3}$$

$$A = \frac{1}{3} \qquad B = \frac{8}{3}$$

$$\int \frac{3x-5}{x^2-x-2}\,dx = \frac{1}{3}\int \frac{1}{x-2}\,dx + \frac{8}{3}\int \frac{1}{x+1}\,dx \; .$$

$$= \frac{1}{3}\ln|x-2| + \frac{8}{3}\ln|x+1| + c$$

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Caso 2: $\Delta = 0$

$$\int \frac{mx+q}{ax^2+bx+c}\,dx$$

$\Delta = 0$ vuol dire che il denominatore $ax^2+bx+c$ ha una sola radice $x_0 = \frac{-b}{2a}$ e

$$ax^2+bx+c = a(x-x_0)^2$$

$$\int \frac{mx+q}{a(x-x_0)^2}\,dx = \frac{1}{a}\int \frac{mx+q}{(x-x_0)^2}\,dx \quad (*)$$

Si cercano $A$ e $B$ t.c.

$$\frac{mx+q}{ax^2+bx+c} = \frac{A}{x-x_0} + \frac{B}{(x-x_0)^2}$$

Una volta trovati $A$ e $B$ si ha che

$$* \quad \frac{1}{a}\left(\int \frac{A}{x-x_0}\,dx + \int \frac{B}{(x-x_0)^2}\,dx\right)$$

$$= \frac{1}{a}\left(A \ln|x-x_0| + B \cdot \left(-\frac{1}{x-x_0}\right)\right) + C$$

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ESEMPIO

$$\int \frac{x}{4x^2+4x+1}\,dx$$

$$\Delta = 16-16=0 \qquad , \; x_0 = \frac{-b}{2a} = \frac{-4}{8} = -\frac{1}{2}$$

$$4x^2+4x+1 = 4\left(x+\frac{1}{2}\right)^2$$

$$\int \frac{x}{4\left(x+\frac{1}{2}\right)^2}\,dx = \frac{1}{4}\int \frac{x}{\left(x+\frac{1}{2}\right)^2}\,dx$$

$$\frac{x}{\left(x+\frac{1}{2}\right)^2} = \frac{A}{x+\frac{1}{2}} + \frac{B}{\left(x+\frac{1}{2}\right)^2} = \frac{A\left(x+\frac{1}{2}\right)+B}{\left(x+\frac{1}{2}\right)^2}$$

$$\underline{x} = A\left(x+\frac{1}{2}\right) + B$$

$$= \underline{A}\,x + \underline{\frac{A}{2} + B}$$

$$\begin{cases} A = 1 \\ \frac{A}{2} + B = 0 \end{cases} \qquad \begin{cases} A = 1 \\ B = -\frac{1}{2} \end{cases}$$

Allora

$$\frac{1}{4}\int \frac{x}{\left(x+\frac{1}{2}\right)^2}\,dx = \frac{1}{4}\left[\int \frac{1}{x+\frac{1}{2}}\,dx - \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2}\,dx\right]$$

$$= \frac{1}{4}\int \frac{1}{x+\frac{1}{2}}\,dx - \frac{1}{8}\int \frac{1}{\left(x+\frac{1}{2}\right)^2}\,dx$$

$$= \frac{1}{4}\ln\left|x+\frac{1}{2}\right| + \frac{1}{8}\,\frac{1}{x+\frac{1}{2}} + C.$$

$$\left[ \int \frac{\overset{f'}{1}}{\underbrace{\left(x+\frac{1}{2}\right)^2}_{f}} dx = -\frac{1}{x+\frac{1}{2}} + C \right]$$

$$\int \frac{1}{x^2+\frac{1}{2}x} dx = \int \frac{1}{x\left(x+\frac{1}{2}\right)} dx$$

cercare $A$, $B$

$$\frac{1}{x\left(x+\frac{1}{2}\right)} = \frac{A}{x} + \frac{B}{x+\frac{1}{2}}$$

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$$\int \frac{4x-5}{x^2-6x+9} dx$$

$$\Delta = 36 - 4 \cdot 9 = 0$$

$$x_0 = \frac{6}{2} = 3$$

$$x^2 - 6x + 9 = (x-3)^2$$

$$\frac{4x-5}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$

$$= \frac{A(x-3) + B}{(x-3)^2}$$

$$4x - 5 = A(x-3) + B$$

$$= Ax - 3A + B$$

$$\begin{cases} A = 4 \\ -3A + B = -5 \end{cases} \rightarrow B = 3A - 5 = 12 - 5 = 7$$

$$\begin{cases} A = 4 \\ B = 7 \end{cases}$$

$$\int \frac{4x-5}{(x-3)^2} dx = \int \frac{4}{x-3} dx + \int \frac{7}{(x-3)^2} dx$$

$$= 4\int \frac{1}{x-3}\,dx + 7\int \frac{1}{(x-3)^2}\,dx$$

$$= 4\ln|x-3| - \frac{7}{x-3} + C.$$

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$\boxed{\Delta < 0}$

$$\int \frac{mx+q}{ax^2+bx+c}\,dx$$

$\Delta<0$ il denominatore non si può fattorizzare. però si può scrivere come somma di quadrati:

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a}$$

se $\Delta<0$ il termine $-\frac{\Delta}{4a^2}$ è positivo.

$$ax^2+bx+c = a\left(\left(x+\frac{b}{2a}\right)^2 \underbrace{-\frac{\Delta}{4a^2}}_{>0}\right)$$

$$= a\left(\left(x+\frac{b}{2a}\right)^2 + \beta^2\right)$$

$D(x) = ax^2+bx+c$

Si cercano $A$ e $B$ t.c.

$$\frac{mx+q}{D(x)} = \frac{A\,D'(x)}{D(x)} + \frac{B}{D(x)}$$

Allora

$$\int \frac{mx+q}{D(x)}\,dx = A\int \frac{D'(x)}{D(x)}\,dx + B\int \frac{1}{D(x)}\,dx$$

$$= A\ln|D(x)| + \frac{B}{a}\int \frac{1}{\left(x+\frac{b}{2a}\right)^2+\beta^2}\,dx$$

$$= A\ln|D(x)| + \frac{B}{a}\,\frac{1}{\beta}\,\arctan\left(\frac{x+\frac{b}{2a}}{\beta}\right) + C.$$

ESEMPIO

$$\int \frac{x+2}{x^2+x+2}\,dx$$

$$\Delta = 1-8=-7<0$$

$$x^2+x+2 = \left(x+\frac{1}{2}\right)^2-\frac{1}{4}+2 = \left(x+\frac{1}{2}\right)^2+\frac{7}{4}$$

$$= \left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2$$

$$x_{1,2} = \frac{-1\pm\sqrt{-7}}{2} = \frac{-1\pm i\sqrt{7}}{2} = -\frac{1}{2}\pm i\frac{\sqrt{7}}{2}$$

<u>Nota</u>

Se $\Delta<0$ e $\alpha\pm i\beta$ sono le radici complesse del polinomio $ax^2+bx+c=0$, allora:

$$ax^2+bx+c = a\left((x-\alpha)^2+\beta^2\right)$$

Nell'esempio $x^2+x+2$:

$$x_{1,2} = \underbrace{-\frac{1}{2}}_{\alpha}\pm i\underbrace{\frac{\sqrt{7}}{2}}_{\beta}$$

$$x^2+x+2 = \left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2$$

Cerchiamo A e B t.c.

$$\frac{x+2}{x^2+x+2} = \frac{A(2x+1)}{x^2+x+2}+\frac{B}{x^2+x+2}$$

$$x+2 = A(2x+1)+B$$

$$= 2Ax+A+B$$

$$\begin{cases} 2A=1 \\ A+B=2 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=2-A=2-\frac{1}{2}=\frac{3}{2} \end{cases}$$

$$\int\frac{x+2}{x^2+x+2}\,dx = \frac{1}{2}\int\frac{2x+1}{x^2+x+2}\,dx+\frac{3}{2}\int\frac{1}{x^2+x+2}\,dx$$

$$= \frac{1}{2}\ln|x^2+x+2|+\frac{3}{2}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\underbrace{\left(\frac{\sqrt{7}}{2}\right)^2}_{\beta}}\,dx$$

$$= \frac{1}{2}\ln|x^2+x+2| + \frac{3}{\cancel{2}}\,\frac{\cancel{2}}{\sqrt{7}}\,\arctan\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)+C$$

$$= \frac{1}{2}\ln|x^2+x+2| + \frac{3}{\sqrt{7}}\arctan\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)+C$$

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ESEMPIO

$$\int \frac{x-3}{x^2+4}\,dx$$

$D(x) = x^2+4$  $\Delta < 0$  $\Delta = 0^2 - 4\cdot 4 = -16$

e $D(x) = x^2 + 2^2$

Cerchiamo $A$, $B$ t.c.

$$\frac{x-3}{x^2+4} = \frac{A(2x)}{x^2+4} + \frac{B}{x^2+4}$$

$$x - 3 = 2Ax + B$$

$$\begin{cases} 2A = 1 \\ B = -3 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \\ B = -3 \end{cases}$$

$$\int \frac{x-3}{x^2+4}\,dx = \frac{1}{2}\int \frac{2x}{x^2+4}\,dx - 3\int \frac{1}{x^2+4}\,dx$$

$$= \frac{1}{2}\ln|x^2+4| - 3\int \frac{1}{x^2+2^2}\,dx$$

$$= \frac{1}{2}\ln(x^2+4) - 3\,\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C$$

$$= \frac{1}{2}\ln(x^2+4) - \frac{3}{2}\arctan\left(\frac{x}{2}\right)+C.$$

Nota

$$\int \frac{3}{x^2+4}\,dx = 3\int \frac{1}{x^2+4}\,dx = 3\int \frac{1}{x^2+2^2}\,dx$$

$$= 3\,\frac{1}{2}\,\arctan\left(\frac{x}{2}\right) + C.$$

$$\frac{3}{x^2+4} = \frac{A\cdot(2x)}{x^2+4} + \frac{B}{x^2+4}$$

$$3 = 2Ax + B$$

$$\begin{cases} A = 0 \\ B = 3 \end{cases}$$

ESEMPIO

$$\int \frac{x}{x^2+2x+5}\,dx$$

$$D(x) = x^2+2x+5 \qquad \Delta = 4 - 4\cdot 5 = -16 < 0$$

$$x^2+2x+5 = x^2+2x+1+4 = (x+1)^2+4$$
$$= (x+1)^2+2^2$$

$$x_{1,2} = \frac{-2\pm\sqrt{-16}}{2} = \frac{-2\pm 4i}{2} = \underbrace{-1}_{\alpha} \pm \underbrace{2}_{\beta}i$$

$$x^2+2x+5 = (x-\alpha)^2+\beta^2 = (x+1)^2+2^2$$

$$\frac{x}{x^2+2x+5} = \frac{A(2x+2)}{x^2+2x+5} + \frac{B}{x^2+2x+5}$$

$$x = 2Ax + 2A + B$$

$$\begin{cases} 2A = 1 \\ 2A + B = 0 \end{cases} \qquad \begin{cases} A = \frac{1}{2} \\ B = -2A = -1 \end{cases}$$

$$\int \frac{x}{x^2+2x+5}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,dx - \int \frac{1}{x^2+2x+5}\,dx$$

$$= \frac{1}{2}\ln|x^2+2x+5| - \int \frac{1}{(x+1)^2+2^2}\,dx$$

$$= \frac{1}{2}\ln|x^2+2x+5| - \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C.$$

1) $\displaystyle\int \frac{x-2}{x^2+x-2}\,dx = -\frac{1}{3}\ln|x-1| + \frac{4}{3}\ln|x+2| + C$

2) $\displaystyle\int \frac{1}{2x^2+x-10}\,dx = \frac{1}{9}\ln|x-2| - \frac{1}{9}\ln|2x+5| + C$

3) $\displaystyle\int \frac{x-1}{x^2+x+1}\,dx = \frac{1}{2}\ln(x^2+x+1) - \sqrt{3}\,\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$

4) $\displaystyle\int \frac{2x-3}{x^2-2x+1}\,dx = 2\ln|x-1| + \frac{1}{x-1} + C$

5) $\displaystyle\int \frac{x-3}{2x^2+x+1}\,dx = \frac{1}{4}\ln|x^2+x+1| - \frac{13}{2\sqrt{7}}\arctan\left(\frac{4x+1}{\sqrt{7}}\right) + C$

Soluzioni:

1) $\displaystyle\int \frac{x-2}{x^2+x-2}\,dx$

$D(x) = x^2+x-2$. $\qquad \Delta = 1+8 = 9$

$$x_{1,2} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2} = \begin{cases} 1 \\ -2 \end{cases}$$

$D(x) = (x-1)(x+2)$

Cerchiamo A e B t.c.

$$\frac{x-2}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$x-2 = Ax+2A+Bx-B$$

$$\begin{cases} A + B = 1 \\ 2A - B = -2 \end{cases}$$

$$\begin{cases} B = 1 - A \\ 2A - 1 + A = -2 \end{cases} \Longleftrightarrow \quad 3A = -1 \Longleftrightarrow A = -\frac{1}{3}$$

$$\begin{cases} B = 1 - \left(-\frac{1}{3}\right) = \frac{4}{3} \\ A = -\frac{1}{3} \end{cases} \Longleftrightarrow \begin{cases} A = -\frac{1}{3} \\ B = \frac{4}{3} \end{cases}$$

$$\int \frac{x-2}{x^2+x-2}\,dx = -\frac{1}{3}\int \frac{1}{x-1}\,dx + \frac{4}{3}\int \frac{1}{x+2}\,dx$$

$$= -\frac{1}{3}\ln|x-1| + \frac{4}{3}\ln|x+2| + C$$

2) $\displaystyle\int \frac{1}{2x^2+x-10}\,dx$

$D(x) = 2x^2 + x - 10$

$\Delta = 1 + 80 = 81$

$$x_{1,2} = \frac{-1 \pm \sqrt{81}}{4} = \frac{-1 \pm 9}{4} = \begin{cases} \frac{8}{4} = 2 \\ \frac{-10}{4} = -\frac{5}{2} \end{cases}$$

$D(x) = 2(x-2)\left(x+\frac{5}{2}\right) = (x-2)(2x+5)$

Cerco $A$ e $B$ t.c.

$$\frac{1}{2x^2+x-10} = \frac{A}{x-2} + \frac{B}{2x+5} = \frac{A(2x+5) + B(x-2)}{(x-2)(2x+5)}$$

$$1 = 2Ax + 5A + Bx - 2B$$

$$\begin{cases} 2A + B = 0 \\ 5A - 2B = 1 \end{cases} \iff \begin{cases} B = -2A \\ 5A + 4A = 1 \end{cases}$$

$$\iff \begin{cases} B = -2A \\ 9A = 1 \end{cases} \iff \begin{cases} A = \frac{1}{9} \\ B = -\frac{2}{9} \end{cases}$$

$$\int \frac{1}{2x^2+x-10}\,dx = \frac{1}{9}\int \frac{1}{x-2}\,dx - \frac{2}{9}\int \frac{1}{2x+5}\,dx$$

$$= \frac{1}{9}\int \frac{1}{x-2}\,dx - \frac{1}{9}\int \frac{2}{2x+5}\,dx$$

$$= \frac{1}{9}\ln|x-2| - \frac{1}{9}\ln|2x+5| + C.$$

3) $\displaystyle\int \frac{x-1}{x^2+x+1}\,dx$

$D(x) = x^2+x+1 \qquad D'(x) = 2x+1$

$\Delta = 1 - 4 = -3 < 0$

Radici complesse $x_{1,2} = \dfrac{-1 \pm \sqrt{-3}}{2} = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Cerco $A$, $B$ t.c.

$$\frac{x-1}{x^2+x+1} = \frac{A(2x+1)}{x^2+x+1} + \frac{B}{x^2+x+1}$$

$$x-1 = A(2x+1) + B$$

$$x-1 = 2Ax + A + B$$

$$\begin{cases} 2A = 1 \\ A+B = -1 \end{cases} \iff \begin{cases} A = \frac{1}{2} \\ B = -1 - A = -1 - \frac{1}{2} = -\frac{3}{2} \end{cases}$$

$$\begin{cases} A = \frac{1}{2} \\ B = -\frac{3}{2} \end{cases}$$

$$\int \frac{x-1}{x^2+x+1}\,dx = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{3}{2}\int \frac{1}{x^2+x+1}\,dx$$

$$= \frac{1}{2}\ln|x^2+x+1| - \frac{3}{2}\int \frac{1}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$(>0 \ \forall x \in \mathbb{R})$

$$= \frac{1}{2}\ln(x^2+x+1) - \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\arctan\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C$$

$$= \frac{1}{2}\ln(x^2+x+1) - \underbrace{\frac{3}{\sqrt{3}}}_{=\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

$$= \frac{1}{2}\ln(x^2+x+1) - \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

4) $\displaystyle\int \frac{2x-3}{x^2-2x+1}\,dx$

$D(x) = x^2-2x+1 \qquad \Delta = 0$

$D(x) = (x-1)^2$

Cerco $A, B$ t.c.

$$\frac{2x-3}{x^2-2x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1)+B}{x^2-2x+1}$$

$2x-3 = A(x-1)+B$

$2x-3 = Ax - A + B$

$$\begin{cases} A = 2 \\ -A+B = -3 \end{cases} \qquad \begin{cases} A = 2 \\ B = -1 \end{cases}$$

$$\int \frac{2x-3}{x^2-2x+1}\,dx = 2\int \frac{1}{x-1} - \int \frac{1}{(x-1)^2}\,dx$$

$$= 2\ln|x-1| - \left(-\frac{1}{x-1}\right) + C$$

$$= 2\ln|x-1| + \frac{1}{x-1} + C$$

5) Soluzione nella prossima lezione: