ESERCITAZ. TRIGONOMETRIA 23/10/2025

Richiami sulle formule trigonometriche

Relaz. fondamentale della trigonometria

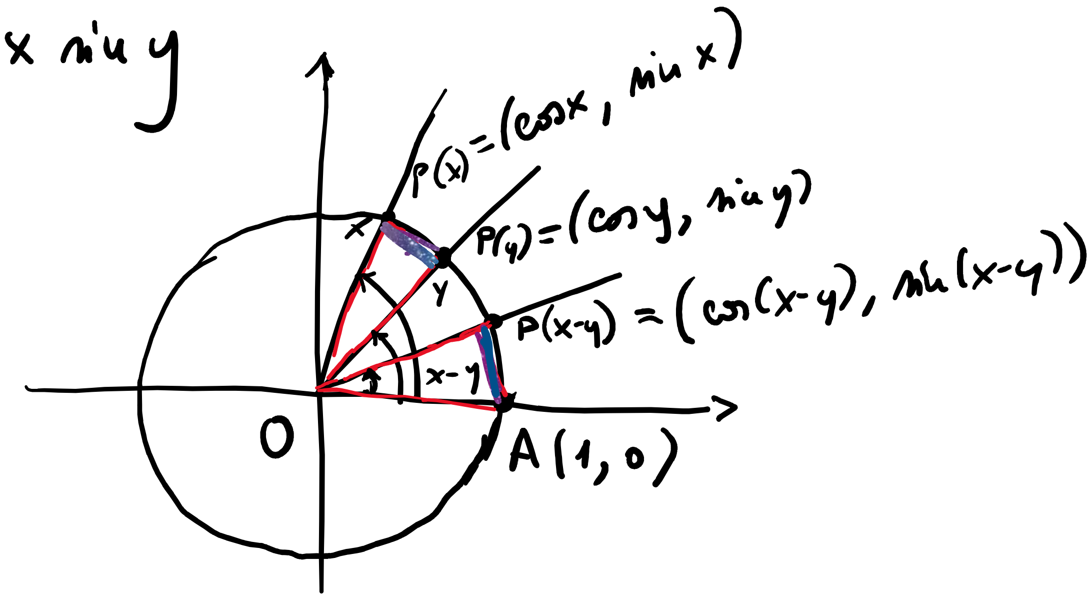
$$\sin^2 x + \cos^2 x = 1 \qquad \forall x \in \mathbb{R}$$

Formule di addizione e sottrazione del seno e coseno

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \forall x, y \in \mathbb{R}$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \qquad \forall x, y \in \mathbb{R}$$

Ad es. dim. la formula

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$



Risulta che i triangoli $\overset{\triangle}{OAP(x-y)}$, $\overset{\triangle}{OP(y)P(x)}$ sono congruenti

$$\Rightarrow \overline{AP(x-y)}^2 = \overline{P(y)P(x)}^2$$

cioè $d^2(A, P(x-y)) = d^2(P(y), P(x))$

$$\Rightarrow (1-\cos(x-y))^2 + \sin^2(x-y) = (\cos x - \cos y)^2 + (\sin x - \sin y)^2$$

$$1 - 2\cos(x-y) + \underbrace{\cos^2(x-y) + \sin^2(x-y)}_{=1} = \cos^2 x + \cos^2 y - 2\cos x \cos y + \sin^2 x + \sin^2 y - 2\sin x \sin y$$

$$\Rightarrow \cancel{2} - 2\cos(x-y) = \cancel{2} - 2\sin x \sin y - 2\cos x \cos y$$

$$\Rightarrow \cos(x-y) = \sin x \sin y + \cos x \cos y.$$

$-\sin y$ $\qquad$ $\cos y$

$\Rightarrow \cos(x-y) = \sin x \sin y + \cos x \cos y .$

$$\cos(x+y) = \cos(x-(-y)) = \sin x \overbrace{\sin(-y)}^{-\sin y} + \cos x \overbrace{\cos(-y)}^{''}$$

$$= \cos x \cos y - \sin x \sin y .$$

archi associati

$$\sin(x+y) \overset{\downarrow}{=} \cos\left(\frac{\pi}{2} - (x+y)\right) = \cos\left(\left(\frac{\pi}{2} - x\right) - y\right) =$$

$$= \cos\left(\frac{\pi}{2} - x\right)\cos y + \sin\left(\frac{\pi}{2} - x\right)\sin y$$

$$= \sin x \cos y + \cos x \sin y .$$

$$\sin(x-y) = \sin(x+(-y)) = \sin x \cos(-y) + \cos x \sin(-y)$$

$$= \sin x \cos y - \cos x \sin y .$$

Formule di duplicazione

$$\begin{cases} \sin(2x) = 2 \sin x \cos x \\ \cos(2x) = \cos^2 x - \sin^2 x \end{cases}$$

si ricavano dalla prec. per $x = y$.

Dalla 2ª possiamo ricavare:

$$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x)$$

$$= 2\cos^2 x - 1 . \quad (1)$$

oppure $\cos 2x = \underset{\smile}{\cos^2 x} - \sin^2 x = 1 - 2\sin^2 x \quad (2)$

$$\Rightarrow \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

possono servire per abbassare il grado di espressioni quadratiche.

Formule di bisezione

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$$

$+$ se $\frac{x}{2}$ appartiene al I o IV quadr.

$-$ se $\frac{x}{2}$ appartiene al II o III quadr.

$$\ldots \frac{\ldots}{2} \qquad \sqrt{\frac{\ldots}{2}}$$

$-$ se $\frac{x}{2}$ appartiene al II o III quadr.

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} \longrightarrow$$

$+$ se $\frac{x}{2}$ appartiene al I e II quadr

$-$ se $\frac{x}{2}$ appartiene al III e IV.

(Si ricavano da (1) e (2))

Formule parametriche

Sia $t = tg\left(\frac{x}{2}\right)$ per $x \neq \pi + 2k\pi$, $k \in \mathbb{Z}$

$\left(\frac{x}{2} \neq \frac{\pi}{2} + k\pi\right)$

Allora

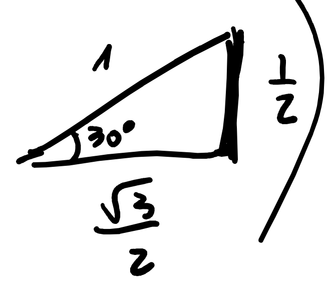
$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad tg\, x = \frac{2t}{1-t^2}$$

Applicaz. delle formule al calcolo di valori "non noti"

Es. $$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}.$$

()

$$\sin\frac{\pi}{2} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) =$$

$$= \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$tg\frac{\pi}{12} = \frac{\sin\frac{\pi}{12}}{\cos\frac{\pi}{12}} = \frac{\frac{\sqrt{6}-\sqrt{2}}{4}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{\sqrt{6}-\sqrt{2}}{\cancel{4}}\cdot\frac{\cancel{4}}{\sqrt{6}+\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$$

$$= \frac{(\sqrt{6}-\sqrt{2})^2}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})} = \frac{6+2-2\sqrt{12}}{6-2} =$$

$$= \frac{8 - 2\sqrt{12}}{4} = \frac{8 - 4\sqrt{3}}{4} = 2 - \sqrt{3}$$

Es. $\cos\frac{\pi}{8} = \cos\left(\frac{1}{2}\cdot\frac{\pi}{4}\right) = + \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}$

$\left(\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}\right) \qquad = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{1}{2}\sqrt{2+\sqrt{2}}$

Es. $\cos\frac{5}{12}\pi = \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \dots$

Formule di prostaferesi. (Trasformano somme in prodotti)

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

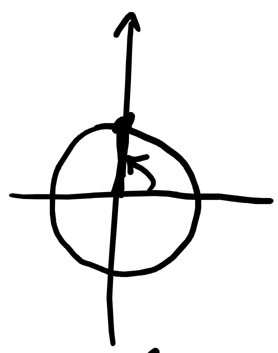
$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

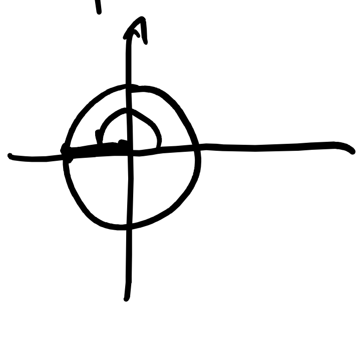
Equazioni e diseq. elementari con le funzioni trigonometriche.

1) $\sin x = 1$

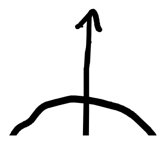


$x = \frac{\pi}{2} + 2k\pi, \quad k\in\mathbb{Z}.$

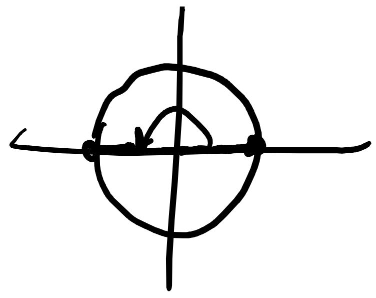
2) $\cos x = -1$



$x = \pi + 2k\pi, \quad k\in\mathbb{Z}$

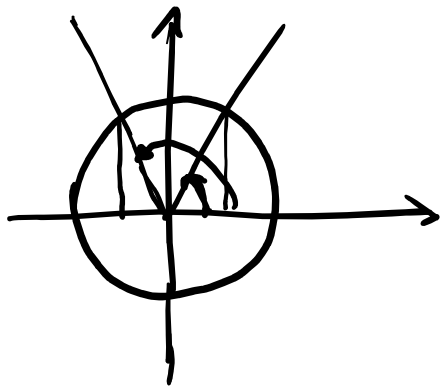


3) $\sin x = 0$



$x = k\pi, \quad k \in \mathbb{Z}$

4) $2 \sin x = \sqrt{3}$

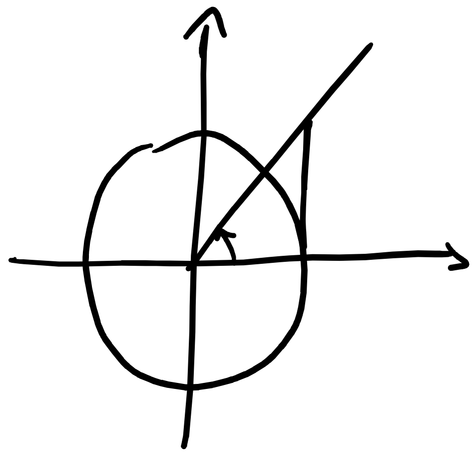


$\sin x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{3} + 2k\pi \quad \vee \quad x = \pi - \frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$

$x = \frac{\pi}{3} + 2k\pi \quad \vee \quad x = \frac{2}{3}\pi + 2k\pi, \quad$ " .

5) $\tan x = 1$



Ricorda:

$$\begin{cases} \tan \frac{\pi}{4} = 1 \\ \tan \frac{\pi}{6} = \frac{\sqrt{3}}{3} \\ \tan \frac{\pi}{3} = \sqrt{3} \end{cases}$$

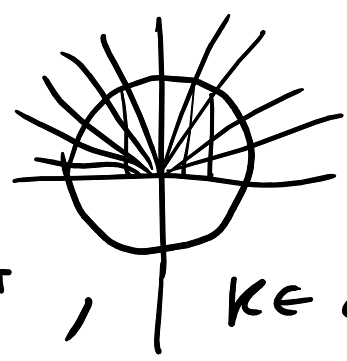
$x = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$

Disequaz. elementari con le funz. trigon.

1) $\sin x > 5 \quad \nexists\, x \in \mathbb{R}$

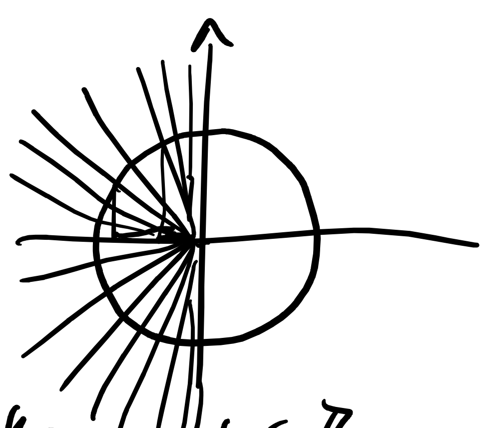
2) $\cos x < -1 \quad \nexists\, x \in \mathbb{R}$

3) $\sin x \geq 0$



$0 + 2k\pi \leq x \leq \pi + 2k\pi, \quad k \in \mathbb{Z}$.

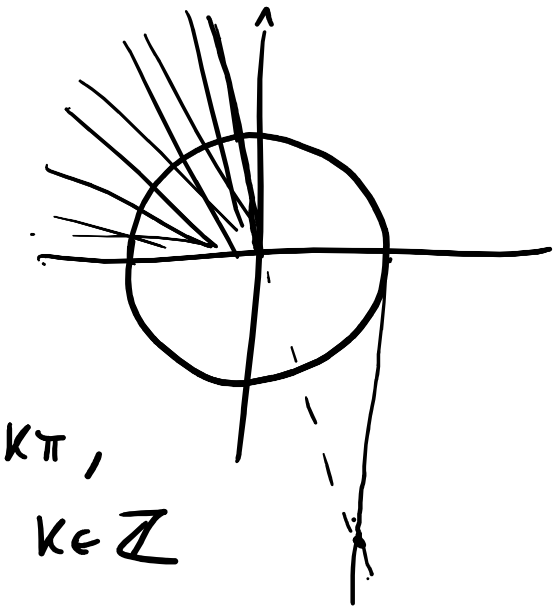
4) $\cos x < 0$

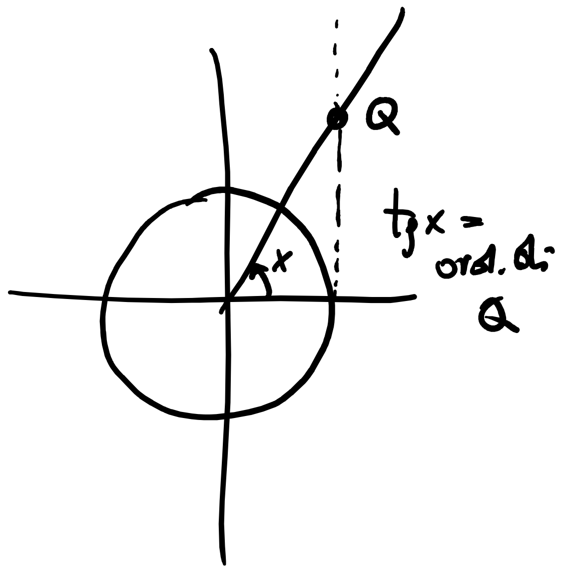


$\frac{\pi}{2} + 2k\pi < x < \frac{3}{2}\pi + 2k\pi, \quad k \in \mathbb{Z}$.

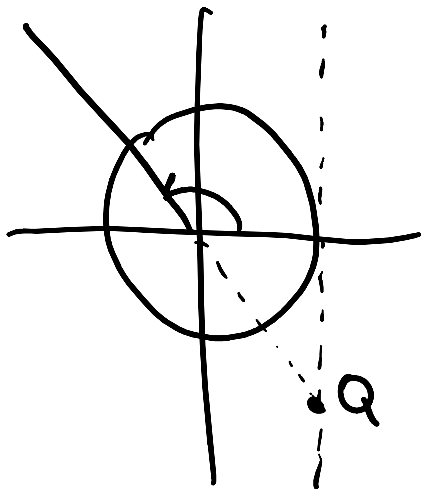
$\frac{\pi}{2} + 2k\pi < x < \frac{3}{2}\pi + 2k\pi, \quad k \in \mathbb{Z}$.

5) $\operatorname{tg} x \leq 0$



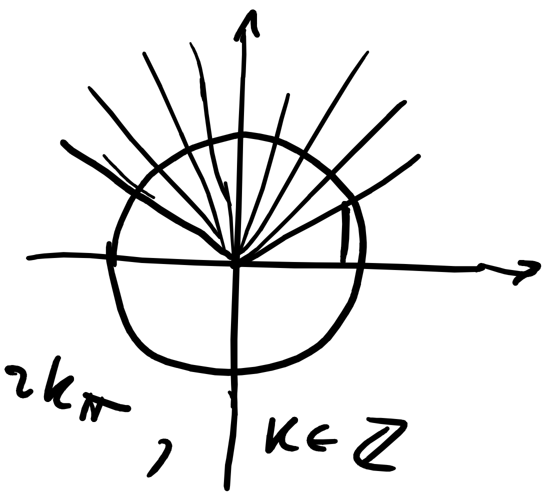


$\frac{\pi}{2} + k\pi < x \leq \pi + k\pi,$
$k \in \mathbb{Z}$



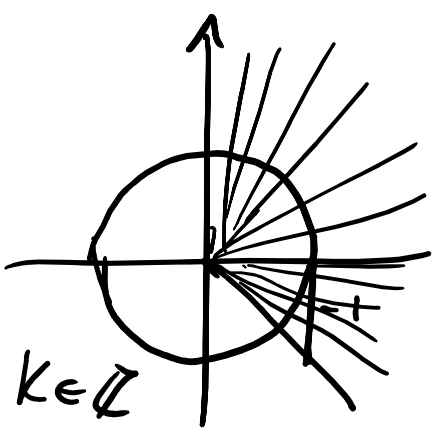
6) $2 \sin x \geq 1$

$\sin x \geq \frac{1}{2}$



$\frac{\pi}{6} + 2k\pi \leq x \leq \frac{5}{6}\pi + 2k\pi, \quad k \in \mathbb{Z}$

7) $\operatorname{tg} x \geq -1$



$-\frac{\pi}{4} + k\pi \leq x < \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$

Eq. e Disequaz. più generali

1) $\sin x + \sin 2x = 0$

$\sin x + 2 \sin x \cos x = 0$

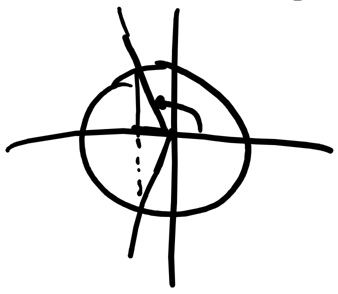
$\sin x \,(1 + 2\cos x) = 0$

$\sin x = 0 \qquad x = k\pi, \; k \in \mathbb{Z}$

$1 + 2\cos x = 0$

$\cos x = -\frac{1}{2}$

$\cos x = -\frac{1}{2}$



$x = \frac{2}{3}\pi + 2k\pi$

$\vee$

$x = -\frac{2}{3}\pi + 2k\pi$

S: $x = k\pi \vee x = \pm \frac{2}{3}\pi + 2k\pi \,, \, k \in \mathbb{Z}$.

2) $\cos^2 x + \sin x = 1$

$\cancel{1} - \sin^2 x + \sin x - \cancel{1} = 0$

$\sin^2 x - \sin x = 0$

$\sin x (\sin x - 1) = 0$

$\sin x = 0 \qquad x = k\pi$

$\vee$

$\sin x = 1 \qquad x = \frac{\pi}{2} + 2k\pi$

S: $x = k\pi \vee x = \frac{\pi}{2} + 2k\pi \,, \, k \in \mathbb{Z}$.

3) $3\cos^2 x - \sin^2 x \leq 0$

$3\cos^2 x - (1 - \cos^2 x) \leq 0$

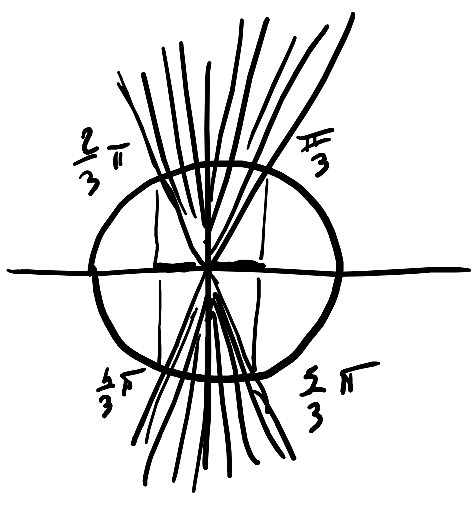
$4\cos^2 x - 1 \leq 0$

$\cos^2 x \leq \frac{1}{4} \qquad t = \cos x$

$t^2 \leq \frac{1}{4}$

$-\frac{1}{2} \leq t \leq \frac{1}{2}$

$-\frac{1}{2} \leq \cos x \leq \frac{1}{2}$



S: $\frac{\pi}{3} + 2k\pi \leq x \leq \frac{2}{3}\pi + 2k\pi$

$\vee \; \frac{4}{3}\pi + 2k\pi \leq x \leq \frac{5}{3}\pi + 2k\pi \,, \quad k \in \mathbb{Z}$

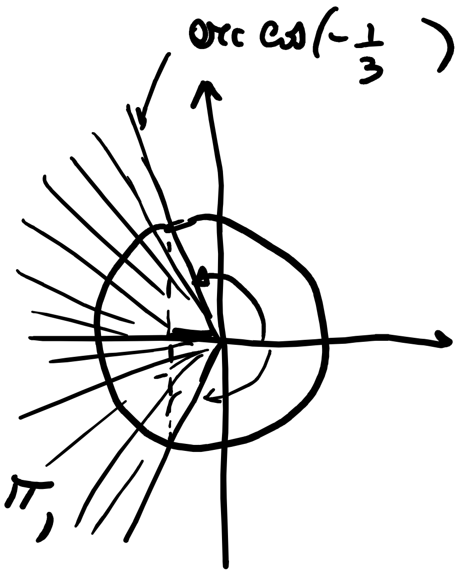
4) $3\cos^2 x - 2\cos x - 1 > 0 \qquad t = \cos x$

$$3t^2 - 2t - 1 > 0$$

$$t_{1/2} = \frac{2 \pm \sqrt{4+12}}{6} = \begin{cases} -\frac{1}{3} \\ 1 \end{cases}$$

$t < -\frac{1}{3} \ \vee \ t > 1$

$\cos x < -\frac{1}{3} \quad \vee \quad \underbrace{\cos x > 1}_{\text{imposs.}}$



$$\arccos\left(-\frac{1}{3}\right) + 2k\pi < x < 2\pi - \arccos\left(-\frac{1}{3}\right) + 2k\pi,$$
$k \in \mathbb{Z}$.

5) $\sin x + \cos x + 1 \le 0$

Usiamo le formule param. $t = tg\frac{x}{2} \qquad x \ne \pi + 2k\pi$

Controlliamo preliminarmente se
$x = \pi + 2k\pi$ è soluz. dell'eq.

$\sin\pi + \cos\pi + 1 = 0 \cancel{-1} + \cancel{1} = 0 \overset{!}{\le} 0$

$\Rightarrow \ x = \pi + 2k\pi$ è soluz. !

Per $x \ne \pi + 2k\pi$, usiamo la sostituz. $t = tg\frac{x}{2}$

$$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 1 \le 0$$

$$\frac{2t+2}{\underbrace{1+t^2}_{>0\,\forall t}} \le 0 \iff t + 1 \le 0$$

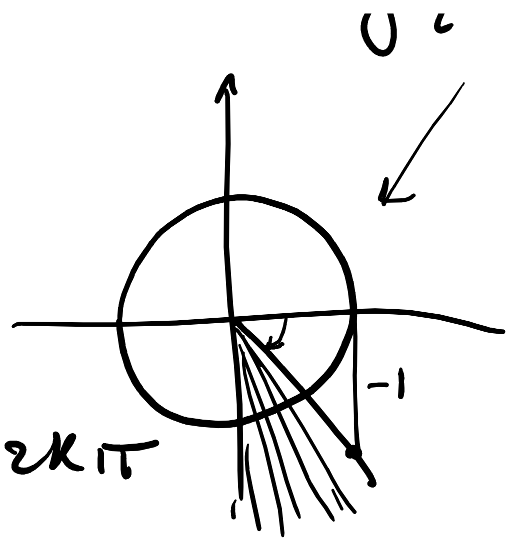
$$t \le -1$$

$$tg\frac{x}{2} \le -1$$

$$-\frac{\pi}{2}+k\pi < \frac{x}{2} \le -\frac{\pi}{4}+k\pi$$

$$\Downarrow$$

$$-\pi+2k\pi < x \le -\frac{\pi}{2}+2k\pi$$



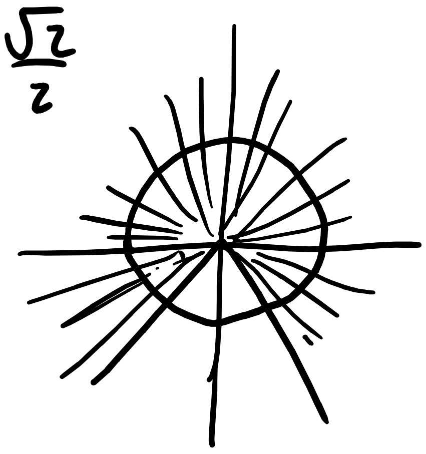
$S: \ -\pi+2k\pi < x \le -\frac{\pi}{2}+2k\pi .$

Diseq. fratta con le funz. trig.

6) $\dfrac{2\sin x+\sqrt{2}}{\cos x} \le 0$

$N \geq 0: \ 2\sin x+\sqrt{2} \geq 0 \qquad \sin x \geq -\frac{\sqrt{2}}{2}$

$$-\frac{\pi}{4} \le x \le \frac{5}{4}\pi$$



$D > 0: \ \cos x > 0$

$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$

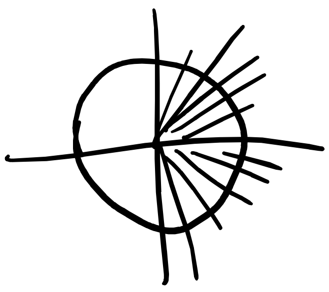
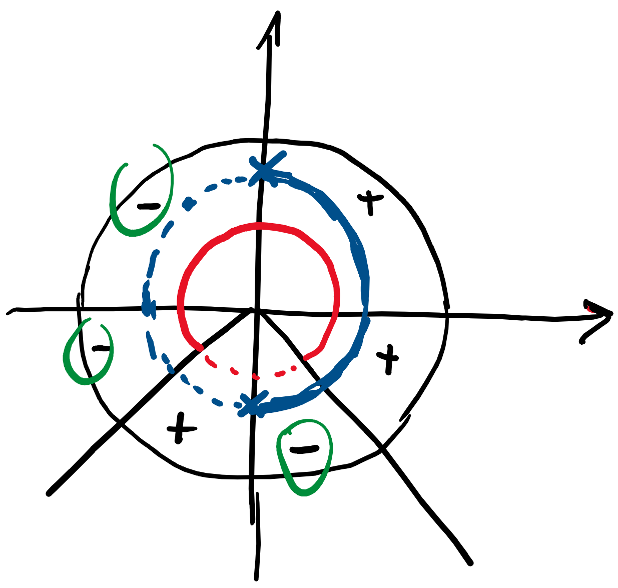


Grafico segni:



$S: \ \frac{\pi}{2} < x \le \frac{5}{4}\pi \ \vee \ \frac{3}{2}\pi < x \le \frac{7}{4}\pi \qquad$ in $[0, 2\pi]$.

Con la periodicità: $\frac{\pi}{2}+2k\pi < x \le \frac{5}{4}\pi+2k\pi \ \vee \ \frac{3}{2}\pi+2k\pi < x \le \frac{7}{4}\pi+2k\pi, \ k\in\mathbb{Z}.$

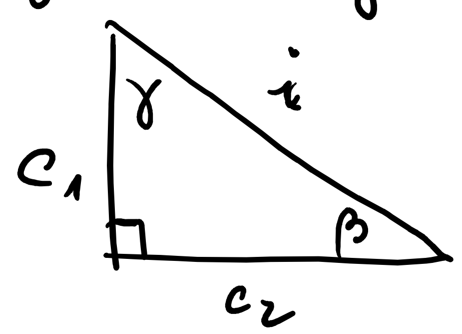
Applicazione della trigonometria alla risoluzione dei triangoli.

Teoremi sui triangoli rettangoli

<u>Teorema</u>: In un triangolo rettangolo, valgono le seguenti:

1) $c_1 = i \operatorname{sen} \beta \qquad c_2 = i \operatorname{sen} \gamma$

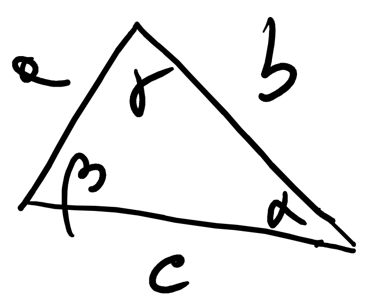
$c_1 = i \cos \gamma \qquad c_2 = i \cos \beta$



2) $\dfrac{c_1}{c_2} = \operatorname{tg} \beta \qquad \dfrac{c_2}{c_1} = \operatorname{tg} \gamma$.

Teoremi sui triangoli qualunque

<u>Teorema dei seni</u>. In un triangolo qualunque, le misure dei lati sono proporzionali ai seni degli angoli opposti, ovvero:

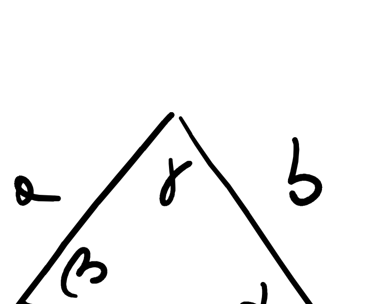


$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

dove α, β, γ sono gli angoli opposti rispettivam. ad a, b, c.

Teor. del <u>coseno</u> (o Teor. di Carnot)

In un triangolo qualunque valgono le seguenti:



$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

dove α, β, γ sono gli angoli rispettivi opposti ad a, b, c.

(Generalizza il Teor. di Pitagora. Infatti, in un triangolo rettangolo: $i^2 = c_1^2 + c_2^2 - \cancel{2c_1c_2 \underbrace{\cos\frac{\pi}{2}}_{=0}}$)