Multiple solutions for a quasilinear Schrödinger equation

Andrzej Szulkin

Joint work with Xiangdong Fang
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Equation

\[ i \partial_t v = \Delta v + W(x)v - \kappa \Delta h(|v|^2)h'(|v|^2)v - f(|v|^2)v \]

models different phenomena in mathematical physics. In one of these models one has \( h(t) = t \) which is the case we consider. For convenience, we take \( \kappa = 1 \).
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We look for standing wave solutions \( v = e^{-i\omega t}u \). Our equation:

\[ -\Delta u + V(x)u - \Delta (u^2)u = g(x, u), \quad x \in \mathbb{R}^N \]
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\[ -\Delta u + V(x)u - \Delta (u^2)u = g(x, u), \quad x \in \mathbb{R}^N \]

Put $G(x, u) := \int_0^u g(x, s) \, ds$. Natural functional:

$$J(u) := \frac{1}{2} \int_{\mathbb{R}^N} (1 + 2u^2)|\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 - \int_{\mathbb{R}^N} G(x, u)$$

$J$ not defined on all of $H^1(\mathbb{R}^N)$ if $N \geq 2$. 

Known methods: constrained minimization (Liu-Wang), reduction to a semilinear case by a change of variables (Liu-Wang-Wang, Colin, Colin-Jeanjean), and very recently, singular perturbation (Liu-Liu-Wang): 

$$J_{\mu}(u) := \mu^4 \int_{\Omega} |\nabla u|^4 + \frac{1}{2} \int_{\mathbb{R}^N} (1 + 2u^2)|\nabla u|^2 + \frac{1}{2} \int_{\Omega} V(x)u^2 - \int_{\mathbb{R}^N} G(x, u)$$

$\Omega \subset \mathbb{R}^N$ bounded, $u \in W^{1,4}_{0}(\Omega)$, $\mu \to 0$. 

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J_\mu(u) := \frac{\mu}{4} \int_{\Omega} |\nabla u|^4 + \frac{1}{2} \int_{\Omega} (1 + 2u^2)|\nabla u|^2
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\[
+ \frac{1}{2} \int_{\Omega} V(x)u^2 - \int_{\Omega} G(x, u),
\]

\( \Omega \subset \mathbb{R}^N \) bounded, \( u \in W^{1,4}_0(\Omega) \), \( \mu \to 0 \).
Our results

(eq1) \(-\Delta u + V(x)u - \Delta (u^2)u = g(x, u), \quad u \in H^1(\mathbb{R}^N)\)

(V) \(V\) is continuous, 1-periodic in \(x_1, \ldots, x_N\) and \(V(x) \geq a_0 > 0\) for all \(x \in \mathbb{R}^N\).

\((g_1)\) \(g\) is continuous, 1-periodic in \(x_1, \ldots, x_N\) and \(|g(x, u)| \leq a(1 + |u|^{p-1}), \) where \(p \in (2, 2 \cdot 2^*)\), \(2^* := 2N/(N - 2)\).

\((g_2)\) \(g(x, u) = o(u)\) uniformly in \(x\) as \(u \to 0\).

\((g_3)\) \(G(x, u)/u^4 \to \infty\) uniformly in \(x\) as \(|u| \to \infty\).

\((g_4)\) \(u \mapsto g(x, u)/u^3\) is positive for \(u \neq 0\), nonincreasing on \((-\infty, 0)\) and nondecreasing on \((0, \infty)\).
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Theorem 1

If (V), (g1)-(g4) are satisfied and \(g\) is odd in \(u\), then (eq1) has infinitely many pairs \(\pm u\) of geometrically distinct solutions.
Let $\mathbb{Z}^N$ act on $H^1(\mathbb{R}^N)$ by $(k \ast u)(x) := u(x - k), \ k \in \mathbb{Z}^N$. Let

$$O(u) := \{k \ast u : k \in \mathbb{Z}^N\}$$

be the orbit of $u$ under this action. By periodicity, if $u$ is a solution of (eq1), then so is $k \ast u$ for any $k \in \mathbb{Z}^N$. Then $O(u)$ is called a critical orbit. Two solutions $u_1, u_2$ are called geometrically distinct if $O(u_1) \cap O(u_2) = \emptyset$. 

A similar result, for a more general LHS, has been proved recently in: Liu, Wang, Guo, Multibump solutions for quasilinear elliptic equations, J. Funct. Anal. 262 (2012), 4040–4102.

Main differences: in Liu et al. $g$ must be differentiable and satisfy an Ambrosetti-Rabinowitz type condition. An example of a function satisfying our conditions but not those in Liu et al. is $g(u) = u^3 \ln(1 + |u|)$. Our argument is different and simpler.
Let $\mathbb{Z}^N$ act on $H^1(\mathbb{R}^N)$ by $(k \ast u)(x) := u(x - k)$, $k \in \mathbb{Z}^N$. Let

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Another equation:

(V) $V$ is continuous, 1-periodic in $x_1, \ldots, x_N$ and $V(x) \geq a_0 > 0$ for all $x \in \mathbb{R}^N$.

(Q) $q$ is continuous, 1-periodic in $x_1, \ldots, x_N$ and $q(x) \geq b_0 > 0$ for all $x \in \mathbb{R}^N$.

(eq2) $-\Delta u + V(x)u - \Delta(u^2)u = q(x)u^3, \quad u \in H^1(\mathbb{R}^N)$
Another equation:

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(eq2) \[-\Delta u + V(x)u - \Delta(u^2)u = q(x)u^3, \quad u \in H^1(\mathbb{R}^N)\]

**Theorem 2**

*If (V) and (Q) are satisfied, then (eq2) has infinitely many pairs \( \pm u \) of geometrically distinct solutions.*
Let $f$ be defined by

$$f(0) = 0, \quad f'(t) = \frac{1}{(1 + 2f^2(t))^{1/2}} \quad \text{for } t \in [0, +\infty)$$

and $f(t) = -f(-t)$ for $t \in (-\infty, 0]$.
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Then \( f(t)/t \to 1 \) as \( t \to 0 \), \( f(t)/\sqrt{t} \to 2^{1/4} \) as \( t \to \infty \).
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Then $f(t)/t \to 1$ as $t \to 0$, $f(t)/\sqrt{t} \to 2^{1/4}$ as $t \to \infty$.

Change of variables: $v := f^{-1}(u)$

New functional:

\[ I(v) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)f^2(v) - \int_{\mathbb{R}^N} G(x, f(v)) \]
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and \( f(t) = -f(-t) \) for \( t \in (-\infty, 0] \).

Then \( f(t)/t \to 1 \) as \( t \to 0 \), \( f(t)/\sqrt{t} \to 2^{1/4} \) as \( t \to \infty \).

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For small \( |v| \), \( f^2(v) \sim v^2 \) and \( G(x, f(v)) = o(v^2) \)

For large \( |v| \), \( f^2(v) \sim |v| \) and \( G(x, f(v))/v^2 \to \infty \).

In the second theorem, \( G(x, u) = \frac{1}{4} q(x)u^4 \), so

\[
G(x, f(v)) \sim q(x)v^2 \quad \text{for large} \quad |v| \quad (I \text{ asymptotically quadratic at 0 and } \infty).
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\[ G(x, f(v)) \sim G(x, \sqrt{|v|}) \] for large \( v \) explains why \( 2 \cdot 2^* \) is the critical exponent for this problem.
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$I \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$

$v$ is a critical point of $I$ if and only if

$$-\Delta v + V(x)f(v)f'(v) = g(x, f(v))f'(v), \quad v \in H^1(\mathbb{R}^N)$$

If $I'(v) = 0$, then $u = f(v) \in H^1(\mathbb{R}^N)$ and $u$ is a solution of our equation.
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If \( l''(v) = 0 \), then \( u = f(v) \in H^1(\mathbb{R}^N) \) and \( u \) is a solution of our equation.

Nehari manifold:

\[ \mathcal{M} := \{ v \in H^1(\mathbb{R}^N) \setminus \{0\} : \langle l''(v), v \rangle = 0 \} \]
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$$\mathcal{M} := \{v \in H^1(\mathbb{R}^N) \setminus \{0\} : \langle I''(v), v \rangle = 0\}$$

$\mathcal{M}$ is unlikely to be of class $C^1$
Let $I \in C^1(E, \mathbb{R})$ (E a Hilbert space), $I(0) = 0$, let $S$ denote the unit sphere and set $\alpha_w(s) := I(sw)$, $w \neq 0$, and suppose for each $w \neq 0$ there exists $s_w$ such that $\alpha'_w(s) > 0$ for $0 < s < s_w$, $\alpha'_w(s) < 0$ for $s > s_w$. There exists $\delta > 0$ such that $s_w \geq \delta$ for all $w \in S$ and $s_w$ is uniformly bounded on compact subsets of $S$. $\alpha'_w(s_w) = \langle I'(s_w)w, w \rangle = 0$, so $s_w \in M$ and $sw \not\in M$ for any other $s > 0$. $m(w) := s_w$, $w \in S$. One shows $m$ is a homeomorphism between $S$ and $M$. $\Psi(w) := I(m(w))$, $w \in S$. 

Modified method of Nehari manifold

(Taken from Weth-Sz, The method of Nehari manifold, Handbook of Nonconvex Analysis and Applications, International Press, Boston, 2010, pp. 597–632)

Let $I \in C^1(E, \mathbb{R})$ ($E$ a Hilbert space), $I(0) = 0$, let $S$ denote the unit sphere and set $\alpha_w(s) := I(sw)$, $w \neq 0$, and suppose

- For each $w \neq 0$ there exists $s_w$ such that $\alpha'_w(s) > 0$ for $0 < s < s_w$, $\alpha'_w(s) < 0$ for $s > s_w$
- There exists $\delta > 0$ such that $s_w \geq \delta$ for all $w \in S$ and $s_w$ is uniformly bounded on compact subsets of $S$
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Let $l \in C^1(E, \mathbb{R})$ ($E$ a Hilbert space), $l(0) = 0$, let $S$ denote the unit sphere and set $\alpha_w(s) := l(sw)$, $w \neq 0$, and suppose

- For each $w \neq 0$ there exists $s_w$ such that $\alpha'_w(s) > 0$ for $0 < s < s_w$, $\alpha'_w(s) < 0$ for $s > s_w$
- There exists $\delta > 0$ such that $s_w \geq \delta$ for all $w \in S$ and $s_w$ is uniformly bounded on compact subsets of $S$

$\alpha'_w(s_w) = \langle l'(s_w w), w \rangle = 0$, so $s_w w \in M$ and $sw \not\in M$ for any other $s > 0$
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- There exists \( \delta > 0 \) such that \( s_w \geq \delta \) for all \( w \in S \) and \( s_w \) is uniformly bounded on compact subsets of \( S \)

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\[ m(w) := s_w w, \quad w \in S \]

One shows \( m \) is a homeomorphism between \( S \) and \( \mathcal{M} \).

\[ \Psi(w) := I(m(w)), \quad w \in S \]
Proposition

(a) \( \Psi \in C^1(S, \mathbb{R}) \) and \( \langle \Psi'(w), z \rangle = \|m(w)\| \langle l'(m(w)), z \rangle \), \( z \in T_w(S) \).

(b) If \( (w_n) \) is a Palais-Smale sequence for \( \Psi \), then \( (m(w_n)) \) is a Palais-Smale sequence for \( l \). If \( (u_n) \subset \mathcal{M} \) is a bounded Palais-Smale sequence for \( l \), then \( (m^{-1}(u_n)) \) is a Palais-Smale sequence for \( \Psi \).

(c) \( w \) is a critical point of \( \Psi \) if and only if \( m(w) \) is a nontrivial critical point of \( l \). Moreover, the corresponding values of \( \Psi \) and \( l \) coincide.

(d) If \( l \) is even, then so is \( \Psi \).
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(b) If \( (w_n) \) is a Palais-Smale sequence for \( \Psi \), then \( (m(w_n)) \) is a Palais-Smale sequence for \( I \). If \( (u_n) \subset M \) is a bounded Palais-Smale sequence for \( I \), then \( (m^{-1}(u_n)) \) is a Palais-Smale sequence for \( \Psi \).

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(d) If \( I \) is even, then so is \( \Psi \).

If \( U \) is an open subset of \( S \), \( sw \notin M \) for any \( s > 0 \) and \( w \in S \setminus U \) and the other assumptions are satisfied on \( U \), then \( \Psi \in C^1(U, \mathbb{R}) \) and the conclusions of Proposition hold on \( U \).

Here \( m \) is a homeomorphism between \( U \) and \( M \).
Proof of Theorem 1

Easy to see that $\alpha_w(s) = I(sw) > 0$ for small $s > 0$
($I(v) \geq a\|v\|^2 + o(\|v\|^2)$ as $v \to 0$).

$\alpha_w(s) = I(sw) \to -\infty$ as $s \to \infty$ (recall $G(x, u)/u^4 \to \infty$
implies $G(x, f(v))/v^2 \to \infty$).

Hence $\max_{s>0} \alpha_w(s)$ exists and is $> 0$.

$u \mapsto g(x, u)/u^3$ nonincreasing on $(-\infty, 0)$, nondecreasing on
$(0, \infty)$ implies (after some work) that $\alpha'_w(s) = 0$ at a unique
$s = s_w > 0$.

So $\Psi(w)$ is well defined and Proposition holds. It is easy to
see that $\inf_S \Psi = c > 0$. Moreover, $\Psi$ inherits the group
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From now on we assume

$\Psi$ has finitely many critical orbits
One shows that $I$ is coercive on $\mathcal{M}$. This implies that if $(w_n)$ is a PS-sequence for $\Psi$ in $\Psi^d := \{\Psi \in S : \Psi \leq d\}$, then $(m(w_n))$ must be bounded.
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**Lemma (Discreteness of PS-sequences)**

If $(w_n^1), (w_n^2) \subset \Psi^d$ are two Palais-Smale sequences for $\Psi$, then either $\|w_n^1 - w_n^2\| \to 0$ or

$$\limsup_{n \to \infty} \|w_n^1 - w_n^2\| \geq \rho(d) > 0,$$

where $\rho(d)$ depends on $d$ but not on the particular choice of Palais-Smale sequences.

Idea goes back to the work of Bartsch-Ding on Palais-Smale attractors.
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Let $K := \{w : \Psi'(w) = 0\}$, $K_d := \{w \in K : \Psi(w) = d\}$. Let $\eta$ be a pseudo-gradient vector field for $\Psi$. 

\[ \eta : \{ (t, w) : w \in S \setminus K, \ T^-(w) < t < T^+(w) \} \to S \setminus K \]

\((T^-(w), T^+(w))\) maximal existence time for \(\eta(\cdot, w)\)

**Lemma**

\[ \lim_{t \to T^+(w)} \eta(t, w) \text{ exists and is a critical point of } \Psi \text{ for each } w \in S \setminus K. \]
\[ \eta : \{(t, w) : w \in S \setminus K, \ T^-(w) < t < T^+(w)\} \rightarrow S \setminus K \]

\((T^-(w), T^+(w))\) maximal existence time for \(\eta(\cdot, w)\)

**Lemma**

\[ \lim_{t \rightarrow T^+(w)} \eta(t, w) \text{ exists and is a critical point of } \Psi \text{ for each } w \in S \setminus K. \]

Since \(\Psi\) is bounded below, a well known argument shows that the limit exists if \(T^+(w) < \infty\), and if this limit were not a critical point, the flow could be continued for \(t > T^+(w)\).

If \(T^+(w) = \infty\) and the limit does not exist, then using boundedness of \(\Psi\) one can construct two PS-sequences \(\eta(t^1_n, w)\) and \(\eta(t^2_n, w)\) which are separated by a distance < \(\rho(d)\), contradicting the discreteness lemma.
Lemma

For every $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that
(a) $\Psi_{d-\varepsilon} \cap K = K_d$ and
(b) $\lim_{t \to T^+(w)} \Psi(\eta(t, w)) < d - \varepsilon$ for $w \in \Psi_{d+\varepsilon} \setminus U_\delta(K_d)$, where $U_\delta(K_d)$ is the open $\delta$-neighbourhood of $K_d$. 
Lemma

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Since there are finitely many critical values, (a) is clearly satisfied.

For (b) one first shows that $\|\Psi'\| \geq \tau > 0$ in $U_{\delta}(K_d) \setminus U_{\delta/2}(K_d)$. Choosing $\varepsilon$ small enough one then shows that if $w \in \Psi_{d-\varepsilon}^{d+\varepsilon} \setminus U_{\delta}(K_d)$ and $\eta(t, w) \in U_{\delta/2}(K_d)$, then $\Psi(\eta(t, w)) < c$. So $\eta(t, w)$ will continue to a critical point below the level $c - \varepsilon$. 
Let

\[ c_k := \inf \{ d \in \mathbb{R} : \gamma(\Psi^d) \geq k \}, \quad k \geq 1 \]

(\( \gamma \) denotes Krasnoselskii’s genus).
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(\( \gamma \) denotes Krasnoselskii’s genus).
Set \( d = c_k \) and \( U = U_\delta(K_d) \).
Since the set \( K_d \) is discrete, \( \gamma(K_d) = 0 \) (if \( K_d = \emptyset \)) or \( \gamma(K_d) = 1 \).
Let
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Since the set \( K_d \) is discrete, \( \gamma(K_d) = 0 \) (if \( K_d = \emptyset \)) or \( \gamma(K_d) = 1 \).
Using the last lemma, we have
\[
\gamma(\Psi^{d+\varepsilon}) \leq \gamma(U) + \gamma(\Psi^{d-\varepsilon}) \leq \gamma(U) + k - 1 = \gamma(K_d) + k - 1
\]
So \( \gamma(K_d) = 1 \) and \( K_d \neq \emptyset \). If \( d \equiv c_k = c_{k+1} \), then \( \gamma(K_d) > 1 \) (because \( \gamma(\Psi^{d+\varepsilon}) \geq k + 1 \)) which is impossible. Hence \( c_{k+1} > c_k \) for all \( k \) and \( \Psi \) has infinitely many critical levels, a contradiction.
Proof of Theorem 2

\[ -\Delta u + V(x)u - \Delta (u^2)u = q(x)u^3, \quad u \in H^1(\mathbb{R}^N) \]

\[ l(v) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)f^2(v) - \frac{1}{4} \int_{\mathbb{R}^N} q(x)f^4(v) \]
Proof of Theorem 2

(eq 2) \(-\Delta u + V(x)u - \Delta(u^2)u = q(x)u^3, \quad u \in H^1(\mathbb{R}^N)\)

\[ I(v) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)f^2(v) - \frac{1}{4} \int_{\mathbb{R}^N} q(x)f^4(v) \]

Let

\[ U := \left\{ w \in S : \int_{\mathbb{R}^N} |\nabla w|^2 < \int_{\mathbb{R}^N} q(x)w^2 \right\} \]

Clearly, \( U \) open in \( S \) and \( U \neq \emptyset \). Recall \( \alpha_w(s) = I(sw) \).
Proof of Theorem 2

(eq2) \[-\Delta u + V(x)u - \Delta(u^2)u = q(x)u^3, \quad u \in H^1(\mathbb{R}^N)\]

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Let

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Clearly, \(U\) open in \(S\) and \(U \neq \emptyset\). Recall \(\alpha_w(s) = I(sw)\).

**Lemma**

1. For each \(w \in U\) there is a unique \(s_w > 0\) such that \(\alpha_w'(s) > 0\) for \(0 < s < s_w\) and \(\alpha_w'(s) < 0\) for \(s > s_w\). Moreover, \(sw \in \mathcal{M}\) if and only if \(s = s_w\).
2. If \(w \notin U\), then \(sw \notin \mathcal{M}\) for any \(s > 0\).
Now \( m(w) := s_w w \) is a homeomorphism between \( U \) and \( M \), and \( \Psi(w) := I(m(w)) \) is of class \( C^1 \) on \( U \).

The proof of Theorem 2 is the same as of Theorem 1 but certain modifications are necessary.
Now $m(w) := s_w w$ is a homeomorphism between $U$ and $\mathcal{M}$, and $\Psi(w) := l(m(w))$ is of class $C^1$ on $U$.

The proof of Theorem 2 is the same as of Theorem 1 but certain modifications are necessary.

Unclear whether $l$ is coercive on $\mathcal{M}$ but one can show that all PS-sequences in $\mathcal{M}$ are bounded. Moreover, if $w_0 \in \partial U$, then $\alpha_{w_0}(s) \equiv l(sw_0) \to \infty$ as $s \to \infty$. Hence, if $w_n \in U$, $w_n \to w_0$, then

$$\alpha_{w_n}(s_{w_n}) = \sup_{s > 0} l(sw_n) = l(s_{w_n}w_n) \to \infty \quad (s_{w_n}w_n \in \mathcal{M}).$$
Now \( m(w) := s_w w \) is a homeomorphism between \( U \) and \( \mathcal{M} \), and \( \Psi(w) := I(m(w)) \) is of class \( C^1 \) on \( U \).

The proof of Theorem 2 is the same as of Theorem 1 but certain modifications are necessary.

Unclear whether \( I \) is coercive on \( \mathcal{M} \) but one can show that all PS-sequences in \( \mathcal{M} \) are bounded. Moreover, if \( w_0 \in \partial U \), then \( \alpha_{w_0}(s) \equiv I(sw_0) \to \infty \) as \( s \to \infty \). Hence, if \( w_n \in U, w_n \to w_0 \), then

\[
\alpha_{w_n}(s_{w_n}) = \sup_{s > 0} I(sw_n) = I(s_{w_n}w_n) \to \infty \quad (s_{w_n}w_n \in \mathcal{M}).
\]

These two facts suffice to show that the flow \( \eta \) with the same properties as in Theorem 1 exists. In particular, since \( I(s_{w_n}w_n) \to \infty \), we see that \( \lim_{t \to T^+(w)} \eta(t, w) \not\in \partial U \), i.e., \( \eta(., w) \) cannot terminate at a point on \( \partial U \).
The proof continues as in the preceding theorem if we can show that $U$ contains sets of arbitrarily large genus.

Since $q \geq b_0 > 0$, it is easy to see using the spectral decomposition of $-\Delta - q$ in $L^2(\mathbb{R}^N)$ that there exists an infinite-dimensional subspace $E_0$ of $H^1(\mathbb{R}^N)$ such that $E_0 \cap S \subset U$. So $\gamma(E_0 \cap S) = \infty$. 
Happy birthday
Anna Maria and Giuliana!
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Buon compleanno
Anna Maria e Giulian

Google's help with this translation is gratefully acknowledged