A Variational Analysis of a Gauged Nonlinear Schrödinger Equation

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Variational and Topological Methods in Nonlinear Phenomena
We are concerned with a planar gauged Nonlinear Schrödinger Equation:

\[ iD_0 \phi + (D_1 D_1 + D_2 D_2) \phi + |\phi|^{p-1} \phi = 0. \]

Here \( t \in \mathbb{R}, \ x = (x_1, x_2) \in \mathbb{R}^2, \ \phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C} \) is the scalar field, \( A_\mu : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \) are the components of the gauge potential, namely \((A_0, A_1, A_2) = (A^0, -A)\), and \( D_\mu = \partial_\mu + iA_\mu \) is the covariant derivative \((\mu = 0, 1, 2)\).
The field equations, in non-relativistic notation, are

\[ \mathbf{B} = \nabla \times \mathbf{A}, \]
\[ \mathbf{E} = -\nabla A^0 - \partial_t \mathbf{A}, \]

where \((\mathbf{E}, \mathbf{B})\) is the electromagnetic field.
The natural and obvious equation of gauge field dynamic is the Maxwell equation:

\[ \partial_\mu F^{\mu\nu} = j^\nu, \]

where

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]

and \(j^\mu\) is the conserved matter current,

\[ j^0 = |\phi|^2, \quad j^i = 2\text{Im} (\bar{\phi} D_i \phi). \]
A modified gauge field equation has been introduced adding the so-called Chern-Simons term into the previous equation [Deser, Hangen, Jackiw, Schonfeld, Templeton, in the ’80s]:

\[ \partial \mu F_{\mu\nu} + \frac{1}{2} \kappa \epsilon_{\nu\alpha\beta} F_{\alpha\beta} = j_{\nu}, \]

where \( \kappa \) is a parameter that measures the strength of the Chern-Simons term, \( \epsilon_{\nu\alpha\beta} \) is the Levi-Civita tensor, and super-indices are related to the Minkowski metric with signature \((1, -1, -1)\). At low energies, the Maxwell term becomes negligible and can be dropped, giving rise to:

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At low energies, the Maxwell term becomes negligible and can be dropped, giving rise to:

\[ \frac{1}{2} \kappa \epsilon^{\nu\alpha\beta} F_{\alpha\beta} = j^{\nu}. \]
Taking for simplicity $\kappa = 2$, we arrive to the system

\[
\begin{aligned}
&iD_0\phi + (D_1D_1 + D_2D_2)\phi + |\phi|^{p-1}\phi = 0, \\
&\partial_0A_1 - \partial_1A_0 = \text{Im}(\bar{\phi}D_2\phi), \\
&\partial_0A_2 - \partial_2A_0 = -\text{Im}(\bar{\phi}D_1\phi), \\
&\partial_1A_2 - \partial_2A_1 = \frac{1}{2}|\phi|^2,
\end{aligned}
\] (1)

where the unknowns are $(\phi, A_0, A_1, A_2)$. 
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&D_0 \phi + (D_1 D_1 + D_2 D_2) \phi + |\phi|^{p-1} \phi = 0, \\
&D_0 A_1 - D_1 A_0 = \text{Im}(\bar{\phi} D_2 \phi), \\
&D_0 A_2 - D_2 A_0 = -\text{Im}(\bar{\phi} D_1 \phi), \\
&D_1 A_2 - D_2 A_1 = \frac{1}{2} |\phi|^2,
\end{align*}
$$

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where the unknowns are $(\phi, A_0, A_1, A_2)$.

As usual in Chern-Simons theory, problem (1) is invariant under gauge transformation,

$$
\phi \to \phi e^{i\chi}, \quad A_\mu \to A_\mu - \partial_\mu \chi,
$$

for any arbitrary $C^\infty$-function $\chi$. 
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The initial value problem, as well as global existence and blow-up, has been addressed in [Bergé, de Bouard & Saut, 1995; Huh, 2009-2013] for the case $p = 3$. 
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$$
\phi(t, x) = u(|x|)e^{i\omega t}, \quad A_0(x) = k(|x|), \\
A_1(t, x) = -\frac{x_2}{|x|^2}h(|x|), \quad A_2(t, x) = \frac{x_1}{|x|^2}h(|x|),
$$

where $\omega > 0$ is a given frequency and $u, k, h$ are real valued functions on $[0, \infty)$ and $h(0) = 0$. 
With this ansatz, (1) becomes

\[
\begin{cases}
-\Delta u + \omega u + A_0u + A_1^2u + A_2^2u - |u|^{p-1}u = 0, \\
\partial_1 A_0 = -A_2 u^2, \\
\partial_2 A_0 = A_1 u^2, \\
\partial_1 A_2 - \partial_2 A_1 = \frac{1}{2} u^2.
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First, they solve $A_0, A_1, A_2$ in terms of $u$. The ansatz implies

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\frac{1}{s} h'(s) = \frac{1}{2} u^2(s).
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\]

From the condition \( h(0) = 0 \), we get

\[
h(r) = \int_0^r \frac{s}{2} u^2(s) \, ds,
\]

and so

\[
A_1(x) = -\frac{x_2}{|x|^2} \int_0^{|x|} \frac{s}{2} u^2(s) \, ds, \quad A_2(x) = \frac{x_1}{|x|^2} \int_0^{|x|} \frac{s}{2} u^2(s) \, ds.
\]
For what concerns $A_0$, we have

$$k'(s) = -\frac{h(s)}{s}u^2(s).$$

By integrating both sides from $r$ to $\infty$, we obtain

$$A_0(x) = k(|x|) = \zeta + \int_{|x|}^{\infty} \frac{h(s)}{s}u^2(s) \, ds,$$

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If $u \in H^1_r(\mathbb{R}^2) \cap C(\mathbb{R}^2)$, then $A_0, A_1, A_2 \in L^\infty(\mathbb{R}^2) \cap C^1(\mathbb{R}^2)$. 
Therefore we need only to solve, in $\mathbb{R}^2$, the equation:

$$-\Delta u + \left( \omega + \xi + \frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s) \, ds \right) u = |u|^{p-1}u.$$
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Let us show that the constant $\omega + \xi$ is a gauge invariant of the stationary solutions of the problem.
Since problem (1) is invariant under gauge transformation,

\[ \phi \rightarrow \phi e^{i\chi}, \quad A_\mu \rightarrow A_\mu - \partial_\mu \chi, \]

for any arbitrary \( C^\infty \) function \( \chi \), if

\[
(\phi, A_0, A_1, A_2) = \left( u(|x|)e^{i\omega t}, k(|x|), -\frac{x_2}{|x|^2}h(|x|), \frac{x_1}{|x|^2}h(|x|) \right),
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is a solution of (1), then, taking \( \chi = ct \),

\[ (\tilde{\phi}, \tilde{A}_0, A_1, A_2) = \left( u(|x|)e^{i(\omega + c)t}, k(|x|) - c, -\frac{x_2}{|x|^2} h(|x|), \frac{x_1}{|x|^2} h(|x|) \right), \]

is a solution of (1), too.
The equation

We will take $\xi = 0$ in what follows, that is,

$$\lim_{|x| \to +\infty} A_0(x) = 0,$$

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Our aim is to solve, in $\mathbb{R}^2$, the nonlocal equation:

$$-\Delta u + \left(\omega + \frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s) \, ds\right) u = |u|^{p-1}u, \quad (\mathcal{P})$$

where

$$h(r) = \int_0^r \frac{s}{2} u^2(s) \, ds.$$
In [Byeon, Huh, Seok, JFA 2012] it is shown that \((P)\) is indeed the Euler-Lagrange equation of the energy functional:

\[
I_\omega : H^1_r(\mathbb{R}^2) \to \mathbb{R},
\]

defined as

\[
I_\omega (u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + \omega u^2) \, dx \\
+ \frac{1}{8} \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \left( \int_0^{|x|} s u^2(s) \, ds \right)^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^2} |u|^{p+1} \, dx.
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Since

$$\frac{1}{|x|^2} \left( \int_0^{|x|} su^2(s) \, ds \right)^2 = \frac{C}{|x|^2} \left( \int_{B(0,|x|)} u^2(x) \, dx \right)^2 \leq C \|u\|_{L^4(\mathbb{R}^2)}^4,$$

then, for any $u \in H^1_{\text{loc}}(\mathbb{R}^2)$,
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then, for any \(u \in H^1_r(\mathbb{R}^2)\),
\[
\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \left( \int_0^{|x|} su^2(s) \, ds \right)^2 \, dx < +\infty.
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then, for any \( u \in H^1_r(\mathbb{R}^2) \),
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\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \left( \int_0^{|x|} su^2(s) \, ds \right)^2 \, dx < +\infty.
\]
Therefore the functional is well defined in \( H^1_r(\mathbb{R}^2) \).
A useful inequality

In [Byeon, Huh & Seok], it is proved that, for any $u \in H^1_r(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} |u(x)|^4 \, dx \leq 2 \left( \int_{\mathbb{R}^2} |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left( \int_0^{|x|} su^2(s) \, ds \right)^2 \, dx \right)^{\frac{1}{2}}.$$
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$$

Furthermore, the equality is attained by a continuum of functions

$$
\left\{ u_l = \frac{\sqrt{8}l}{1 + |lx|^2} \in H^1_r(\mathbb{R}^2) : l \in (0, +\infty) \right\}.
$$
Byeon-Huh-Seok results: case $p > 3$

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- This problem is bypassed by using a constrained minimization taking into account the Nehari and Pohozaev identities, in the spirit of [Ruiz, JFA 2006] for the Schrödinger-Poisson equation.
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- This problem is bypassed by using a constrained minimization taking into account the Nehari and Pohozaev identities, in the spirit of [Ruiz, JFA 2006] for the Schrödinger-Poisson equation.
- Infinitely many solutions have been found in [Huh, JMP 2012] for $p > 5$ (possibly sign-changing): this case is more easy since (PS)-condition holds.
Byeon-Huh-Seok results: case $p = 3$

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Any standing wave solutions $(\phi, A_0, A_1, A_2)$ of the previous type with $|\phi| > 0$ have the following form:

$$(\phi, A_0, A_1, A_2) = \left( \frac{\sqrt{8}le^{i\omega t}}{1 + |l|x|^2}, \left( \frac{2l}{1 + |l|x|^2} \right)^2 - \omega, \frac{-2l^2 x_2}{1 + |l|x|^2}, \frac{2l^2 x_1}{1 + |l|x|^2} \right),$$

where $l > 0$ is an arbitrary real constant.
Byeon-Huh-Seok results: case $1 < p < 3$

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By the gauge invariance, this is not a problem if we are looking for solutions $(\phi, A_0, A_1, A_2)$ of the entire system (1).
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For what concerns the single equation $(\mathcal{P})$, a solution $u$ is found only for a particular value of $\omega$. 
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The global behavior of the energy functional $I_\omega$ is not studied.
Analogy with Schrödinger-Poisson equation?

The nonlinear Schrödinger-Poisson equation is
\[\begin{align*}
-\Delta u + u + \lambda \phi u &= |u|^{p-1}u \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^3,
\end{align*}\]

(2)

If \( \bar{u} \) is a critical point of the functional
\[I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx + \lambda \frac{4}{3} \int_{\mathbb{R}^3} (|x|^2 |\bar{u}|^2 \bar{u}^2) \, dx - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \, dx,
\]

and if \( \phi \bar{u} = \frac{1}{2} |x|^2 |\bar{u}|^2 \),

then \((\bar{u}, \phi \bar{u})\) is a solution of (2).
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here \( \lambda > 0 \).

If \( \bar{u} \) is a critical point of the functional

\[
\mathcal{I}_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx + \frac{\lambda}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |u|^2 \right) u^2 \, dx
\]

\[
- \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \, dx,
\]

and if

\[
\phi_{\bar{u}} = \frac{1}{|x|} * |\bar{u}|^2,
\]

then \((\bar{u}, \phi_{\bar{u}})\) is a solution of (2).
Analogy with Schrödinger-Poisson equation?

First difference: while the “critical” exponent for problem (\(\mathcal{P}\)) is \(p = 3\), for (2) it is \(p = 2\).
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The case $1 < p < 3$ for problem $(P)$ is similar to the case $1 < p < 2$ for (2)?
Analogy with Schrödinger-Poisson equation?

First difference: while the “critical” exponent for problem \( \mathcal{P} \) is \( p = 3 \), for (2) it is \( p = 2 \).

The case \( 1 < p < 3 \) for problem \( \mathcal{P} \) is similar to the case \( 1 < p < 2 \) for (2)?

In particular, in [Ruiz, JFA 2006] it is proved that if \( 1 < p < 2 \), for \( \lambda \) small enough, the functional \( \mathcal{I}_\lambda \) is bounded from below and it possesses at least two critical points: one is the minimum, at negative level, the other is a mountain-pass critical point, at positive level.
On the boundedness from below of $I_\omega$

Let $p \in \left(1, \frac{3}{2}\right)$. There exists $\omega_0$ such that:

- if $\omega \in \left(0, \omega_0\right)$, then $I_\omega$ is unbounded from below;
- if $\omega = \omega_0$, then $I_{\omega_0}$ is bounded from below, not coercive and $\inf I_{\omega_0} < 0$;
- if $\omega > \omega_0$, then $I_\omega$ is bounded from below and coercive.

$\omega_0$ has an explicit expression:

$$\omega_0 = 3 - \frac{p}{3} + \frac{p}{3} - \frac{1}{2} \left(3 - p\right)^2 \left(3 + p\right)^2 - \frac{p}{2} \left(3 - p\right)^2,$$

with $m = \int_{-\infty}^{+\infty} \left(2p + 1 \cosh 2 \left(p - \frac{1}{2} r\right)\right)^2 \frac{1}{1 - p} dr$. 

On the boundedness from below of $I_\omega$

Theorem (A.P. & D. Ruiz)

Let $p \in (1, 3)$. There exists $\omega_0$ such that:

- if $\omega \in (0, \omega_0)$, then $I_\omega$ is unbounded from below;

$\omega_0$ has an explicit expression:

$$\omega_0 = 3 - \frac{p}{3} + \frac{p}{3-p} - \frac{1}{2} \left( \frac{3}{3-p} \right)^2 - \frac{1}{2} \left( \frac{3}{3-p} \right)^{1/2} \left( \frac{2p+1}{2} \right) \left( \frac{1}{2} \right)^{1/2} \left( \frac{3}{3-p} \right)^{1/2}.$$

$$m = \int_{-\infty}^{\infty} \left( 2p+1 \right) \cosh \left( \frac{1}{2} \right)^{1/2} \left( \frac{3}{3-p} \right)^{1/2} \left( \frac{2p+1}{2} \right)^{1/2} \left( \frac{1}{2} \right)^{1/2} \left( \frac{3}{3-p} \right)^{1/2} \right) \left( \frac{2p+1}{2} \right)^{1/2} \left( \frac{1}{2} \right)^{1/2} \left( \frac{3}{3-p} \right)^{1/2} \right).$$
Theorem (A.P. & D. Ruiz)

Let $p \in (1, 3)$. There exists $\omega_0$ such that:

- if $\omega \in (0, \omega_0)$, then $I_\omega$ is unbounded from below;
- if $\omega = \omega_0$, then $I_{\omega_0}$ is bounded from below, not coercive and $\inf I_{\omega_0} < 0$;

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$$\omega_0 = 3 - \frac{p}{3} + \frac{p}{3} - \frac{1}{2} \left( 3 - p \right)^2 - p - \frac{1}{2} \left( 3 - p \right)$$

with $m = \int_{-\infty}^{+\infty} \left( 2p + 1 \cosh^2 \left( \frac{p - 1}{2} r \right) \right) \frac{1}{1 - p} dr$. 
On the boundedness from below of $I_\omega$

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On the boundedness from below of $I_ω$

**Theorem (A.P. & D. Ruiz)**

Let $p \in (1, 3)$. There exists $ω_0$ such that:

- if $ω \in (0, ω_0)$, then $I_ω$ is unbounded from below;
- if $ω = ω_0$, then $I_{ω_0}$ is bounded from below, not coercive and $\inf I_{ω_0} < 0$;
- if $ω > ω_0$, then $I_ω$ is bounded from below and coercive.

$ω_0$ has an explicit expression:

$$ω_0 = \frac{3 - p}{3 + p} \left( \frac{p^2 - 1}{2(3 - p)} \right)^{\frac{1}{2 - p}} \left( \frac{m^2(3 + p)}{p - 1} \right)^{-\frac{p - 1}{2(3 - p)}}$$

with

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Rough sketch of the proof

- $I_\omega$ is coercive when the problem is posed on a bounded domain.
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- If unbounded, the sequence $u_n$ behaves as a soliton, if $u_n$ is interpreted as a function of a single real variable.
- $I_\omega$ admits a natural approximation through a limit functional.
- The critical points of that limit functional, and their energy, can be found explicitly, so we can find $\omega_0$. 
Let $u$ a fixed function, and define $u_{\rho}(r) = u(r - \rho)$. Let us now estimate $I_\omega(u_{\rho})$ as $\rho \to +\infty$. 
The limit functional

Let \( u \) a fixed function, and define \( u_\rho(r) = u(r - \rho) \). Let us now estimate \( I_\omega(u_\rho) \) as \( \rho \to +\infty \).

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(2\pi)^{-1}I_\omega(u_\rho) = \frac{1}{2} \int_0^{+\infty} (|u'_\rho|^2 + \omega u^2_\rho) r \, dr \\
+ \frac{1}{8} \int_0^{+\infty} u^2_\rho(r) \left( \int_0^r s u^2_\rho(s) \, ds \right)^2 \, dr \\
- \frac{1}{p + 1} \int_0^{+\infty} |u_\rho|^{p+1} r \, dr.
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It is natural to define the limit functional $J_\omega : H^1(\mathbb{R}) \to \mathbb{R}$,

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Of course

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We will show

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Let $p \in (1, 3)$ and $\omega > 0$. Then:

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a) $J_\omega$ is coercive and attains its infimum;

b) $0$ is a local minimum of $J_\omega$;

c) there exists $\omega_0 > 0$ such that $\min J_\omega < 0$ if and only if $\omega \in (0, \omega_0)$. 
The limit problem

The critical points of $J_\omega$ are solutions of the nonlocal equation

$$- u'' + \omega u + \frac{1}{4} \left( \int_{-\infty}^{+\infty} u^2(s) \, ds \right)^2 u = |u|^{p-1} u, \quad \text{in } \mathbb{R}. \quad (\mathcal{P}_\infty)$$
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Observe that $(\mathcal{P}_\infty)$ can be written as

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One dimensional nonlinear Schrödinger equation

For any $k > 0$, let $w_k \in H^1(\mathbb{R})$ be the unique positive radial solution of:

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Any solution of (3) is of the form $u(r) = \pm w_k(r - \xi)$, for some $\xi \in \mathbb{R}$. 

Moreover, $w_k(r) = k^{1/p - 1/2} w_1(\sqrt{kr})$.

In what follows we define $m := \int_{-\infty}^{+\infty} w_k^2 \, dr$. 

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- if $\omega > \omega_1$, (4) has no solution and there is no nontrivial solution of $(P_\infty)$;
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Moreover

\[
\omega_1 = \left( \frac{(5-p)m^2}{4(p-1)} \right)^{-\frac{p-1}{2(3-p)}} - \frac{m^2}{4} \left( \frac{(5-p)m^2}{4(p-1)} \right)^{-\frac{5-p}{2(3-p)}}.
\]
The map $\psi$

Let us evaluate $J_\omega$ on the curve $k \mapsto w_k$. We have

$$\psi(k) = J_\omega(w_k) = m \left[ \frac{p - 5}{2(3 + p)} k^{\frac{3+p}{2(p-1)}} + \frac{\omega}{2} k^{\frac{5-p}{2(p-1)}} + \frac{m^2}{24} k^{\frac{3(5-p)}{2(p-1)}} \right].$$
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\frac{d}{dk} \psi(k) = m \ k^{\frac{7-3p}{2(p-1)}} \frac{5 - p}{4(p-1)} \left[ -k + \omega + \frac{1}{4} m^2 k^{\frac{5-p}{p-1}} \right].
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The roots of (4) are exactly the critical points of \( \psi \).

Since

\[
\frac{5 - p}{2(p - 1)} < \frac{3 + p}{2(p - 1)} < \frac{3(5 - p)}{2(p - 1)},
\]

\( \psi \) is increasing near 0 (for \( \omega > 0 \)) and near infinity.
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The map $\psi$

- If $\omega > \omega_1$, $\psi$ is positive and increasing without critical points.
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- If $\omega = \omega_0$, $\psi(k_2) = 0$. Observe then, in this case, the minimum of $J_{\omega_0}$ is 0, and is attained at 0 and $w_{k_2}$.
- If $\omega \in (0, \omega_0)$, $\psi(k_2) < 0$ and then $w_{k_2}$ is the unique global minimizer of $J_\omega$ and $J_\omega(w_{k_2}) < 0$. 
The threshold value $\omega_0$

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In order to get the value of $\omega_0$, observe that $J_{\omega_0}(w_{k_2}) = 0$.

Therefore, $\omega_0 > 0$ solves:

$$
\begin{cases}
\frac{d}{dk} \psi(k) = 0, \\
\psi(k) = 0.
\end{cases}
$$

and we infer that

$$
\omega_0 = \frac{3 - p}{3 + p} 3^{\frac{p-1}{2(3-p)}} 2^{\frac{2}{3-p}} \left( \frac{m^2(3 + p)}{p - 1} \right)^{-\frac{p-1}{2(3-p)}}.
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The values \( \omega_0(p) < \omega_1(p) \), for \( p \in (1, 3) \).

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For some specific values of $p$, $m$ can be explicitly computed, and hence $\omega_0$ and $\omega_1$. For instance, if $p = 2$, $m = 6$, $\omega_1 = \frac{2}{9\sqrt{3}}$ and $\omega_0 = \frac{2}{5\sqrt{15}}$. 
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For some specific values of $p$, $m$ can be explicitly computed, and hence $\omega_0$ and $\omega_1$. For instance, if $p = 2$, $m = 6$, $\omega_1 = \frac{2}{9\sqrt{3}}$ and $\omega_0 = \frac{2}{5\sqrt{15}}$. More in general, we have
On the boundedness from below of $I_\omega$

Theorem (A.P. & D. Ruiz)

Let $p \in (1, 3)$. We have:

- if $\omega \in (0, \omega_0)$, then $I_\omega$ is unbounded from below;
- if $\omega = \omega_0$, then $I_{\omega_0}$ is bounded from below, not coercive and $\inf I_{\omega_0} < 0$;
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- if $\omega > \omega_0$, then $I_\omega$ is bounded from below and coercive.

Since $J_\omega(w_{k_2}) < 0$, if $\omega \in (0, \omega_0)$, and

$$I_\omega(w_{k_2}(\cdot - \rho)) \sim 2\pi \rho J_\omega(w_{k_2}), \text{ as } \rho \to +\infty,$$

the first part is proved.
Sketch of the proof: case $\omega \geq \omega_0$
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- There exists $u_n$ a minimizer for $I_\omega|_{H^1_{0,r}(B(0,n))}$. Moreover,

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  \[ I_\omega(u_n) \to \inf I_\omega, \text{ as } n \to +\infty. \]
- If $u_n$ is bounded, then $I_\omega(u_n)$ is also bounded and therefore $\inf I_\omega$ is finite. In what follows we assume that $u_n$ is an unbounded sequence.
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- If $u_n$ is interpreted as a function of a single real variable, $u_n$ does not vanish.
Sketch of the proof: case $\omega \geq \omega_0$

- Let $\xi_n \in \mathbb{R}$ be the largest “center of mass” of $u_n$: we have that $\xi_n \sim \|u_n\|^2$. 
Sketch of the proof: case $\omega \geq \omega_0$

- Let $\bar{\zeta}_n \in R$ be the largest “center of mass” of $u_n$: we have that $\bar{\zeta}_n \sim \|u_n\|^2$.
- Define $\psi_n : [0, +\infty] \to [0, 1]$ be a smooth function such that

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\psi_n(r) = \begin{cases} 
0, & \text{if } r \leq \bar{\zeta}_n - 3\|u_n\|, \\
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  \end{cases}
  \]
- Let’s estimate $I_\omega(u_n)$ with $I_\omega(\psi_n u_n)$ and $I_\omega ((1 - \psi_n)u_n)$:
  \[
  I_\omega(u_n) \geq I_\omega(u_n \psi_n) + I_\omega(u_n(1 - \psi_n))
  + c\|u_n(1 - \psi_n)\|_{L^2(\mathbb{R}^2)}^2 + O(\|u_n\|).
  \]
Sketch of the proof: conclusion for $\omega > \omega_0$

Since

$$I_\omega(u_n \psi_n) = 2\pi \xi_n J_\omega(u_n \psi_n) + O(\|u_n\|).$$
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  \]

- Since $\|u_n \psi_n\|_{H^1(\mathbb{R})} \geq c$, we can prove that $J_\omega(u_n \psi_n) \to c > 0$. 
Sketch of the proof: conclusion for $\omega > \omega_0$

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Since $\xi_n \sim \|u_n\|^2$, we infer that $I_\omega(u_n) > I_\omega(u_n(1 - \psi_n)).$
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- Since $\|u_n \psi_n\|_{H^1(\mathbb{R})} \geq c$, we can prove that $J_\omega(u_n \psi_n) \to c > 0$.
- Since $\xi_n \sim \|u_n\|^2$, we infer that $I_\omega(u_n) > I_\omega(u_n(1 - \psi_n))$.
- Contradiction with the definition of $u_n$, which proves that $\inf I_\omega > -\infty$. 

Sketch of the proof: conclusion for $\omega > \omega_0$

- Let us now show that $I_\omega$ is coercive.
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- Let us now show that $I_\omega$ is coercive.
- Take $u_n \in H^1_r(\mathbb{R}^2)$ an unbounded sequence, and assume that $I_\omega(u_n)$ is bounded from above.
- $\|u_n\|_{L^2(\mathbb{R}^2)} \to +\infty$.
- Then for any $\omega_0 < \omega' < \omega$,

$$I_{\omega'}(u_n) = I_\omega(u_n) + \frac{\omega' - \omega}{2} \|u_n\|^2_{L^2(\mathbb{R}^2)} \to -\infty,$$

a contradiction, since we know that $I_{\omega'}$ is bounded from below.
Sketch of the proof: conclusion for \( \omega = \omega_0 \)
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- We reach a contradiction unless $J_{\omega_0}(u_n\psi_n) \to 0$. 
Sketch of the proof: conclusion for $\omega = \omega_0$

- We reach a contradiction unless $J_{\omega_0}(u_n \psi_n) \to 0$.
- This implies that $\psi_n u_n (\cdot - \xi_n) \to \omega_{k_2}$, where $\omega_{k_2}$ is the nontrivial minimum of $J_{\omega_0}$.
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- With this extra information, we have a better estimate:

\[
I_{\omega_0}(u_n) \geq 2\pi \xi_n J_{\omega_0}(u_n \psi_n) + I_{\omega_0}(u_n(1 - \psi_n)) \\
+ c\|u_n(1 - \psi_n)\|_{L^2(\mathbb{R}^2)}^2 + O(1).
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$$I_{\omega_0}(u_n) \geq I_{(\omega_0 + 2c)}(u_n (1 - \psi_n)) + O(1).$$
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- We already know that $I_{(\omega_0 + 2c)}$ is bounded from below, and hence $\inf I_{\omega_0} > -\infty$. 
On the solutions of $(\mathcal{P})$
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**Theorem (On the boundedness of $I_\omega$)**

Let $p \in (1,3)$. We have:

- if $\omega \in (0,\omega_0)$, then $I_\omega$ is unbounded from below;
- if $\omega = \omega_0$, then $I_{\omega_0}$ is bounded from below, not coercive and $\inf I_{\omega_0} < 0$;
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**Theorem (A.P. & D. Ruiz)**

- For almost every $\omega \in (0,\omega_0]$, $(\mathcal{P})$ admits a positive solution.

Moreover, there exist $\bar{\omega} > \tilde{\omega} > \omega_0$ such that:

- if $\omega > \bar{\omega}$, then $(\mathcal{P})$ has no solutions different from zero;
- if $\omega \in (\omega_0,\tilde{\omega})$, then $(\mathcal{P})$ admits at least two positive solutions: one of them is a global minimizer for $I_\omega$ and the other is a mountain-pass solution.
On the solutions of \((\mathcal{P})\)

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Sketch of the proof: case $\omega \in (0, \omega_0]$
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Performing the rescaling $u \mapsto u_\omega = \sqrt{\omega} \, u(\sqrt{\omega} \cdot)$, we get

$$I_\omega(u_\omega) = \omega \left[ \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) \, dx + \frac{1}{8} \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \left( \int_0^{|x|} su^2(s) \, ds \right)^2 \, dx ight. $$

$$\left. - \frac{\omega^{p-3}}{p+1} \int_{\mathbb{R}^2} |u|^{p+1} \, dx \right].$$
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The geometrical assumptions of the Mountain Pass Theorem are satisfied and we can apply the monotonicity trick [Struwe, Jeanjean], finding a solution for almost every $\omega \in (0, \omega_0]$. 
Sketch of the proof: case $\omega > \omega_0$

Recall that, for any $u \in H^1_r(\mathbb{R}^2)$,

$$
\int_{\mathbb{R}^2} |u(x)|^4 \, dx
\leq 2\left( \int_{\mathbb{R}^2} |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left( \int_0^{|x|} s u^2(s) \, ds \right)^2 \, dx \right)^{\frac{1}{2}}.
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Let $u$ be a solution of $(\mathcal{P})$. We multiply $(\mathcal{P})$ by $u$ and integrate. By this inequality, we get

$$0 \geq \frac{1}{4} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} \left( \omega u^2 + \frac{3}{4} u^4 - |u|^{p+1} \right) \, dx.$$
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\]

Since there exists $\bar{\omega} > 0$ such that, for $\omega > \bar{\omega}$, the function $t \mapsto \omega t^2 + \frac{3}{4} t^4 - |t|^{p+1}$ is non-negative, then $u = 0$. 
Sketch of the proof: case $\omega > \omega_0$

Since $\inf I_{\omega_0} < 0$, there exists $\tilde{\omega} > \omega_0$ such that $\inf I_{\omega} < 0$ for $\omega \in (\omega_0, \tilde{\omega})$.

Being $I_{\omega}$ coercive and weakly lower semicontinuous, the infimum is attained (at negative level). If $\omega \in (\omega_0, \tilde{\omega})$, the functional satisfies the geometrical assumptions of the Mountain Pass Theorem.

Since $I_{\omega}$ is coercive, (PS) sequences are bounded. We find a second solution (a mountain-pass solution) which is at a positive energy level.
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